Charles University in Prague

Faculty of Mathematics and Physics

MASTER THESIS

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Generace vířivosti rychlostního pole gradientem entropie
Generation of vorticity in velocity field by entropy gradient

Mathematical Institute of Charles University

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, August 6, 2012 Signature

Název práce: Generace vířivosti rychlostního pole gradientem entropie

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Abstrakt: V této práci je studován vliv gradientu entropie na vířivost rychlostního pole především využitím rovnic bilance hybnosti. Tyto rovnice jsou formulovány pro Termo-viskózní tekutinu (dále jen tekutinu) a pro Termo-visko-elastický materiál (dále jen pevné látka) společně se zbylými bilancemi (energie, hmotnosti...). K odvození těchto bilancí je použit přístup klasické mechaniky kontinua a zároveň Variační princip téhož. V práci je kladen důraz na přístup pomocí Variačního principu, jenž je modifikací Batemanova principu [Bat29] a na jeho srovnání s klasickým přístupem, právě v případě generace vířivosti, kde klasický přistup je spojován s prací L. Crocca [Cro37]. Je ukázáno, že zavedením dissipativní entropie s_{dis} je dosaženo shody obou přístupů a ve vhodné limitě je ukázán přímý vliv gradientu entropie na vířivost rychlostního pole. Za pomoci tohoto závěru je ukázán vztah mezi změnou cirkulace kolem uzavřené křivky a uvolněným teplem na zvolené geometrii.

Klíčová slova: Vířivost; entropie; dissipace; tření.

Title: Generation of vorticity in velocity field by entropy gradient

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Abstract: The master thesis studies the impact of the entropy gradient on the vorticity of velocity field, particularly by applying the linear momentum balances. These balances are formulated for Thermo-viscous fluids (later fluids) and Thermo-viscous-elastic material (later solids) with the other balances (energy, mass, etc.). In order to derive these balances, the Classical continuum mechanics approach is used along with the respective Variational principles. The thesis emphasizes the Variational principles application representing the modification of the Bateman principle [Bat29] and its comparison with the Classical approach, linked to the L. Crocco work [Cro37], particularly in the case of vorticity generation. It is pointed that by the definition of the dissipative entropy s_{dis} a harmony of both approaches can be achieved and that, in the case of an appropriate limit, the direct effect of the entropy gradient on the vorticity of velocity field can be demonstrated. By applying this conclusion the relationship between the change of circulation among a closed curve and released heat on the given geometry is indicated.

Keywords: Vorticity; entropy; dissipation; friction.

Contents

List of Symbols

Notation

$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$	tensor in the Reference configuration
$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$	tensor in the Current configuration
$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$	vector in the Reference configuration
$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$	vector in the Current configuration
$\boldsymbol{X}$	Lagrangian coordinates
$\boldsymbol{x}$	Eulerian coordinates
$A, B, C, \ldots$	scalar in the Reference configuration
$a, b, c, \ldots$	scalar in the Current configuration
t	time

Operators

$\lvert a \rvert$	absolute value of a
$\overline{\mathfrak{a}}^{dev}$	deviatoric part of a tensor $\mathfrak{a}$
$\nabla_{\boldsymbol{X}} \cdot ()$	divergence in the Reference configuration
$\nabla_{\boldsymbol{x}} \cdot ()$	divergence in the Current configuration (If it is clear, just the $\nabla \cdot ()$ is used)
$\dot{a}$	general derivative of a, see (2.7)
ϵ_{ijk}	tensor Levi-Civita
$\nabla_{\boldsymbol{\chi}} ()$	gradient with respect to trajectory, i.e. $\frac{\partial}{\partial \boldsymbol{\chi}}$
$\nabla_{\boldsymbol{X}} ()$	gradient in the Reference configuration, i.e. $\frac{\partial}{\partial \boldsymbol{X}}$
$\nabla_{\boldsymbol{x}} ()$	gradient in the Current configuration,, i.e. $\frac{\partial}{\partial \boldsymbol{x}}$ (If it is clear, just the $\nabla ()$ is used)
$\mathbb{I}, \delta_{ij}$	identity, e.g. Kroenecer's delta
$\left(\frac{\partial a}{\partial b}\right)_{\mathrm{c}}$	Partial derivative of a by b with constant c
$\Delta ()$	laplace
$\dot{\overline{a}}$	material derivative, $\dot{\overline{a}} = [a]_{,\mathrm{t}} + v_i \frac{\partial a}{\partial x_i}$
$[a]_{,\mathrm{t}}$	partial derivative of a by time t, $\frac{\partial a}{\partial t} = [a]_{,\mathrm{t}}$
$\frac{\partial a}{\partial b}$	partial derivative of a by b
$\nabla_{\boldsymbol{X}} \times ()$	rotation in the Reference configuration

$\nabla_{\boldsymbol{x}} \times ()$	rotation in the Current configuration (If it is clear, just the $\nabla \times ()$ is used)
$\frac{\mathrm{d}a}{\mathrm{d}b}$	total derivative of a by b
$\mathrm{tr}\,(\mathfrak{a})$	trace of tensor $\mathfrak{a}$, $\mathrm{tr}\,(\mathfrak{a}) = \mathfrak{a} : \mathbb{I} = a_{ij}\delta_{ij}$
d	differential
δ	variation
$\wedge$	Outer product, for his definition see (3.11)

Variables

$\mathscr{A}$	Action
Γ	circulation
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
$\mathcal{L}$	density of Lagrangian
s	entropy
ϵ_0	energy
ϵ_T	total energy
$\boldsymbol{f}$	external forces
Δc_w^{w+a}	humidity
$\delta\boldsymbol{\varphi}$	infinitesimal rotation
L	Lagrangian
c_w^{w+a}	mass concentration
$\dot{\overline{\boldsymbol{\beta}}}$	friction force
$\mathcal{V}$	volume of system in the Reference configuration
ν	volume of system in the Current configuration
Φ	potential of volume forces
ω	scalar potential of the potential part of velocity field $\boldsymbol{v}^{\mathrm{pot}}$
$\boldsymbol{\alpha}$	vector potential of the rotational part of velocity field $\boldsymbol{v}^{\mathrm{rot}}$
ρ	density of mass (density)
s_{dis}	dissipative entropy
s_{eq}	equilibrium entropy
s_{ext}	extended entropy
$\mathfrak{e}$	deformation tensor
$\mathfrak{t}^{\mathrm{dis}}$	dissipative part of the stress tensor
$\mathfrak{t}^{\mathrm{el}}$	elastic part of the stress tensor
$\mathfrak{t}$	stress tensor
T	temperature

$\boldsymbol{\chi}^{-1}$	trajectory in the Reference configuration, $\boldsymbol{\chi}^{-1} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, \mathrm{t})$
$\boldsymbol{\chi}$	trajectory in the Current configuration, $\boldsymbol{\chi} = \boldsymbol{\chi}(\boldsymbol{X}, \mathrm{t})$
$\boldsymbol{\beta}$	friction velocity
$\boldsymbol{\varepsilon}$	infinitesimal displacement
$\boldsymbol{q}$	heat flux
$\boldsymbol{v}$	velocity field (velocity)
$\boldsymbol{w}$	vorticity
$\boldsymbol{v}^{\mathrm{pot}}$	potential part of velocity field
$\boldsymbol{v}^{\mathrm{rot}}$	rotational part of velocity field
$\boldsymbol{q}, q$	generalized coordinate
$\dot{\boldsymbol{q}}, \dot{q}$	generalized velocity
h	specific enthalpy (enthalpy)
h_T	total enthalpy
H_{evap}	evaporated heat
l	specific density of Lagrangian
m	amount mass of a system (weight)
p	pressure

Chapter 1

Introduction

The thesis will study the continuum mechanics balances in the Eulerian description from the classical point of view and under the scope of Variational principles, where the consequences between these two insights will be emphasized.

By the Classical mechanics the approach based on the general balance law for quantity $\mathbf{\Phi}$ will be assumed, where its change will be considered as a consequence of its production $\mathcal{P}(\mathbf{\Phi})$ in the control volume ν and its flux $\mathcal{J}(\mathbf{\Phi})$ into the volume.

In order to specify the quantities which have to be balanced and also to outline the notation of the Variational principles the result of the elegant Lagrangian mechanics of *particles* motion will be summarized. This will be achieved by an accent to the physical groundings of the theory, as it was made by Landau and Lifschitz [LL76].

After these preliminary assumptions, we will be able to formulate the variational principle for Thermo-viscous fluids (later fluids), which, as will be shown, follows the Bateman's principle.

Bateman can be taken as the first successful founder on the field of the Variational principles of continuum mechanics. By assuming the Clebsch representation of velocity [Cle09], Bateman [Bat29] derived the equations of motion of ideal fluid in barotropic limit. After this success, there were many attempts to generalize this theory to the case of compressible fluid flows, but the first successful was made by Herivel [Her55], who was able to describe the motion of ideal fluid flow with no barotropic restriction. However, similarly to the Bateman's variational principle, his study approached only irrotational flows. The involvement of rotational flows was made by Lin [LY87], who introduced a new set of constrains, which were added using the Lagrangian multipliers. The main disadvantage of the so called Herivel - Lin's principle were the Lagrangian multipliers, which led to the Clebsch's velocity representation involving 8 potentials, where the assignment of the physical meaning remains difficult until now.

A great summarization of these results along with some interesting predictions were made by Seliger and Whitham [SW67]. It should be noted that these were all the results for *inviscid-isentropic* fluid flow.

The efforts of the physical interpretation on the field of the Variational continuum fluid mechanics was one of the motivations to derive the balances of motion for the viscous fluids flow, in which the physical interpretation would be clear. The derivation for fluids was, among others, achieved in the author's Bachelor thesis [Nov10], where, unfortunately, a sufficient space for the necessary commentary was not left. So this work can be taken as the completion of the Bachelor thesis in this task.

Further the variational principle will be generalized on the case of Thermo-viscous-elastic material (solids) in an effort to show that the difference between the results of the chosen Variational principle and the results of the Classical approach, which will also be derived, belongs to the nature of the chosen Variational principle. The viscosity (in general all irreversible processes) can be introduced by an appropriate entropy production term, the form of which can be modified by means of the terms arising from the commonly accepted balance equations.

Finally, the direct relation between the vorticity generation and entropy gradient will be derived and the new insight to the classical Crocco theorem [Cro37] will be discussed. Moreover, this relation will enable to explain the circulation generation by the chemical reactions and phase transitions as it is observed, e.g. at twisters and hurricanes.

Chapter 2

Lagrangian formalism

Among classical theory Lagrangian formalism presents the most elegant way to deal with the description of motion of a *particle*, or generally particles generating the mechanical system.

During a motion of such mechanical system, which is variable with time, it is needed to specify (describe) $2s$ variables, in order to cover the *degrees of freedom* of mechanical system. To do so, generalized coordinates $\boldsymbol{q} = (q_1, \ldots, q_s)$ and generalized velocities $\dot{\boldsymbol{q}} = (\dot{q}_1, \ldots, \dot{q}_s)$ in Lagrangian formalism are used.

Classical approach to the description of motion evolution are the *Lagrange's equations of the second kind*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = 0, \tag{2.1}$$

where $L = L(q_1, \ldots, q_\mathrm{s}, \dot{q}_1, \ldots, \dot{q}_\mathrm{s}, \mathrm{t}) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mathrm{t})$ is a definite *characteristic* function of generalized coordinate $\boldsymbol{q} = (q_1, \ldots, q_\mathrm{s})$ and velocities $\dot{\boldsymbol{q}} = (\dot{q}_1, \ldots, \dot{q}_s)$ called *Lagrangian*. However, in the following part it will be shown that these equations are a special form of Hamilton's variational principle, which is more convenient for generalizations which will be made in the thesis.

2.1 Hamilton's variational principle

The main assumption of the most general formulation governing the motion of mechanical system is that every mechanical system with the *Lagrangian*

$$L = L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \mathrm{t}),$$

is evolving from the time t_1 and coordinate $\boldsymbol{q}^{(1)}$ to the time t_2 and coordinate $\boldsymbol{q}^{(2)}$, so that the functional

$$\mathscr{A} = \int_{t_1}^{t_2} L\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right)\, \mathrm{d}t \tag{2.2}$$

called *action* takes the least possible value. Thence the action becomes extremal value, the Minimum.

Let us introduce the condition to minimum of the action functional. Let, for simplicity, the system has just one degree of freedom, which is defined by the scalar function $q(\mathrm{t})$. And let this function $q = q(\mathrm{t})$ is the one, for which action is minimal. This means that action $\mathscr{A}$ is increased when $q(\mathrm{t})$ is replaced by any function in the form

$$q(\mathrm{t}) + \delta q(\mathrm{t}),$$

where $\delta q(\mathrm{t})$ is a function which is small everywhere in the interval of time from t_1 to t_2. $\delta q(\mathrm{t})$ is called a *variation* of the function $q(\mathrm{t})$.

Since we consider that all the function for the time $t = t_1$ and $t = t_2$ must take the values $\boldsymbol{q}^{(1)}$ and $\boldsymbol{q}^{(2)}$ respectively, it follows that the variations for these times must be equal to zero [1]

$$\delta q(\mathrm{t}_1) = \delta q(\mathrm{t}_2) = 0. \tag{2.3}$$

Let us observe the change of the action $\mathscr{A}$ when $q(\mathrm{t})$ is replaced by $q(\mathrm{t}) + \delta q(\mathrm{t})$

$$\delta\mathscr{A} \equiv \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q})\,\mathrm{d}t - \int_{t_1}^{t_2} L(q, \dot{q}, t)\,\mathrm{d}t. \tag{2.4}$$

When this difference is expanded just in first powers of variables in the integrand, the function $\delta\mathscr{A}$ is called *first action variation*. From the necessary condition on the minimum of the functional we get that this first action variation, or simply action variation, should be zero. Thus the Hamilton principle may be written in the form

$$\delta\mathscr{A} = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\,\mathrm{d}t = 0. \tag{2.5}$$

This can be rewritten by using the variation

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right)\,\mathrm{d}t = 0. \tag{2.6}$$

Considering

$$\delta\dot{q} = \frac{\mathrm{d}\delta q}{\mathrm{d}t} \tag{2.7}$$

and using Green's theorem, we obtain

$$\delta\mathscr{A} = \left[\frac{\partial L}{\partial \dot{q}}\delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q\,\mathrm{d}t = 0. \tag{2.8}$$

The condition of fixed ends (2.3) implies that first term in right hand side equals zero. The remaining integral has to equal zero for all variations δq. It could be fulfilled only if the integrand is identically equal to zero. That implies the *Lagrange's equation of the second kind*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0. \tag{2.9}$$

If we consider more than one degree of freedom, let say s, the s different functions $q_i(\mathrm{t})$ have to be varied independently in the Hamilton formalism. From the linearity of variations we obtain the *Lagrange's equations of the second kind*, as has been said.

[1] This condition is called *fixed ends*

2.2 Conservation laws

In the case of Lagrangian formalism, the term *conservation law* is mostly equivalent to the term *integral of the motion*. Functions are called integrals of motion if their value remains constant during the motion and if they depend just on their initial conditions. The main importance lies in the integrals of motion that are based on the homogeneity and isotropy of space and time.

2.2.1 Homogeneity of time

At first we consider the law of conservation, which is the consequence of time homogeneity. If closed a system does not explicitly depend on time, then the Lagrangian L is not dependent on time either. It implies that

$$L = L(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{2.10}$$

If we apply the total time derivative on the Lagrangian in the form (2.10), we get

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\ddot{\boldsymbol{q}}. \tag{2.11}$$

When we consider Lagrange's equation (2.1) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\dot{\boldsymbol{q}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0 \tag{2.12}$$

with the context of (2.11), together we obtain

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\dot{\boldsymbol{q}} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\ddot{\boldsymbol{q}} = \tag{2.13}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}}\right). \tag{2.14}$$

From this relation directly implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}} - L\right) = 0. \tag{2.15}$$

Therefore, the parameter E defined by

$$E \equiv \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}} - L \tag{2.16}$$

is conserved during the motion of the closed system. This parameter, which we call *energy*, is the integral of motion, which corresponds to the homogeneity of time.

If we realize that general Lagrangian of closed system has the shape[2] of

$$L = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q}), \tag{2.17}$$

where *T* is the kinetic energy and *U* is potential energy, thence we obtain from (2.16) an important relation for energy

$$E = \frac{\partial\left(T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q})\right)}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}} - T(\boldsymbol{q}, \dot{\boldsymbol{q}}) + U(\boldsymbol{q}) = \tag{2.18}$$

$$= 2T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - T(\boldsymbol{q}, \dot{\boldsymbol{q}}) + U(\boldsymbol{q}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) + U(\boldsymbol{q}), \tag{2.19}$$

assuming that kinetic energy is a quadratic function of the velocities.

[2]Much more general commentary is placed in book written by Landau and Lifshitz [LL76]

2.2.2 Homogeneity of space

The second law of conservation derives from space homogeneity. Under this assumption, mechanical properties of a closed system are unchanged by any parallel displacement[3] of the entire system in space. At this moment we consider such infinitesimal displacement $\boldsymbol{\varepsilon}$.

A change of Lagrangian L, which is caused by this infinitesimal displacement, is

$$\delta L \equiv \sum_\alpha \left(L(\boldsymbol{r}_\alpha + \delta\boldsymbol{r}_\alpha) - L(\boldsymbol{r}_\alpha)\right) = \sum_\alpha \frac{\partial L}{\partial \boldsymbol{r}_\alpha}\delta\boldsymbol{r}_\alpha = \boldsymbol{\varepsilon}\sum_\alpha \frac{\partial L}{\partial \boldsymbol{r}_\alpha}, \tag{2.20}$$

where $\boldsymbol{r}_\alpha$ is a radius vector of the particle α, the velocity of which we have taken as constant.

From space homogeneity we get

$$\frac{\partial L}{\partial \boldsymbol{r}_\alpha} = 0 \qquad \forall\alpha = 1, 2, \dots, \tag{2.21}$$

where the infinitesimal displacement is arbitrary. Lagrange's equations (2.1) give us a condition that

$$\sum_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{v}_\alpha}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_\alpha \frac{\partial L}{\partial \boldsymbol{v}_\alpha}\right) = 0. \tag{2.22}$$

This relation implies that in closed mechanical systems the vector $\boldsymbol{P}$ is conserved, which is defined below

$$\boldsymbol{P} \equiv \sum_\alpha \frac{\partial L}{\partial \boldsymbol{v}_\alpha}, \tag{2.23}$$

and is called *Linear momentum*.

As has been said, kinetic energy is a quadratic function of velocities, of which it is known even more. Kinetic energy has a shape of

$$T = \frac{1}{2} m_\alpha \boldsymbol{v}_\alpha^2.$$

This property in the sense of relation (2.23) gives us

$$\boldsymbol{P} = \sum_\alpha m_\alpha \boldsymbol{v}_\alpha. \tag{2.24}$$

From the last relation we can see the additivity of *Linear momentum* of the system. Moreover, unlike the energy, the momentum of the system is equal to the sum of its values $\boldsymbol{p}_\alpha = m_\alpha \boldsymbol{v}_\alpha$ for the individual particles, whether or not the interaction between them can be neglected. It should be remembered that *all* the three components of the momentum are conserved only in the case of absence of the external field. The individual Linear momentum may be conserved even in the presence of a field, even if the potential in the field does not depend on all three *Cartesian coordinates*[4].

[3] Parallel displacement is such transformation, in which every particle in the system is shifted by the same amount in the same direction, i.e. the radius vector $\boldsymbol{r}$ becomes radius vector $\boldsymbol{r} + \boldsymbol{\varepsilon}$.

[4] The mechanical properties of the system are unchanged by a displacement along the axis of a coordinate which does not the potential energy. Thence the corresponding component of *linear momentum* is conserved.

In case of describing a motion in generalized coordinates, a derivative of Lagrangian by generalized velocities

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \tag{2.25}$$

is called *generalized momentum* and similarly a derivative by generalized coordinates

$$\boldsymbol{F} = \frac{\partial L}{\partial \boldsymbol{q}} \tag{2.26}$$

is called *generalized force*.

This notation implies that

$$\dot{\boldsymbol{p}} = \boldsymbol{F}. \tag{2.27}$$

It should be mentioned, that in the Cartesian coordinates the generalized momentum are the components of the vectors $\boldsymbol{p}_\alpha$, but in general the p_i are linear homogeneous functions of the generalized velocities $\dot{q}_i$.

To conclude, from this section, it should be seen that homogeneity of space stands behind *the law of conservation of total momentum of the closed mechanical system*.

2.2.3 Isotropy of space

Let us derive what conservation law follows the *isotropy of space*. This isotropy means that mechanical properties of the closed system do *not* vary when the system is rotated as a whole in arbitrary manner in space. Let us, therefore, consider an infinitesimal rotation of the system $\delta\boldsymbol{\varphi}$, and search for the condition for the Lagrangian to remain unchanged during the rotation.

A magnitude of the rotation $\delta\boldsymbol{\varphi}$ is the angle of rotation $\delta\varphi$, and its direction is determined by the axis of rotation[5].

At first we should find a resulting increment $\delta\boldsymbol{r}$ in the radius vector $\boldsymbol{r}$ from an origin on the axis to any particle in the system, which is undergoing rotation. The relation between them is quite clear from the picture 2.1, thus

$$\delta\boldsymbol{r} = \delta\boldsymbol{\varphi} \times \boldsymbol{r}. \tag{2.28}$$

If the system is rotated, a direction of the particles velocities are changing too and all vectors are transformed in the same manner like by radius vector. An increment of velocity $\delta\boldsymbol{v}$ relative to a fixed system of coordinates is

$$\delta\boldsymbol{v} = \delta\boldsymbol{\varphi} \times \boldsymbol{v}. \tag{2.29}$$

Let us consider a variation of the Lagrangian

$$\begin{aligned}\delta L &= \sum_\alpha L(\boldsymbol{r}_\alpha + \delta\boldsymbol{r}_\alpha, \boldsymbol{v}_\alpha + \delta\boldsymbol{v}_\alpha) - L(\boldsymbol{r}_\alpha, \boldsymbol{v}_\alpha) = \\ &= \sum_\alpha \left(\frac{\partial L}{\partial \boldsymbol{r}_\alpha}\delta\boldsymbol{r}_{\boldsymbol{\alpha}} + \frac{\partial L}{\partial \boldsymbol{v}_\alpha}\delta\boldsymbol{v}_{\boldsymbol{\alpha}} \right)\end{aligned} \tag{2.30}$$

[5]The direction of rotation is that of a *right-handed* screw driven along $\delta\boldsymbol{\varphi}$.

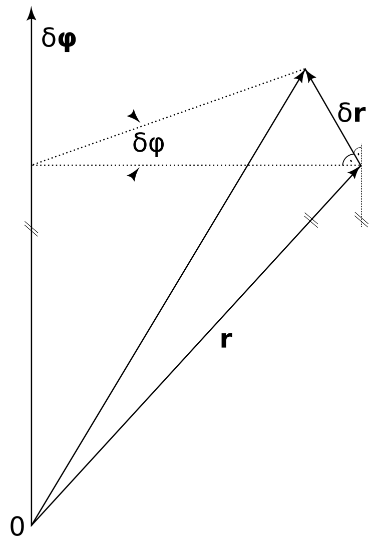

Figure 2.1: Geometry used for the description of the system

which equals zero from space isotropy. When we substitute the derivative $\frac{\partial L}{\partial \boldsymbol{v}_\alpha}$ by $\boldsymbol{p}_\alpha$ and $\frac{\partial L}{\partial \boldsymbol{r}_\alpha}$ by $\dot{\boldsymbol{p}}_\alpha$, we get

$$\sum_\alpha \left(\dot{\boldsymbol{p}}_\alpha \cdot \delta\boldsymbol{\varphi} \times \boldsymbol{r}_\alpha + \boldsymbol{p}_\alpha \cdot \delta\boldsymbol{\varphi} \times \boldsymbol{v}_\alpha\right) = 0. \tag{2.31}$$

By permuting the factors in the last result and taking $\delta\boldsymbol{\varphi}$ outside the sum, we reach

$$\delta\boldsymbol{\varphi} \sum_\alpha \left(\boldsymbol{r}_\alpha \times \dot{\boldsymbol{p}}_\alpha + \boldsymbol{v}_\alpha \times \boldsymbol{p}_\alpha\right) = \tag{2.32}$$

$$= \delta\boldsymbol{\varphi} \sum_\alpha \left(\boldsymbol{r}_\alpha \times \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p}_\alpha) + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_\alpha) \times \boldsymbol{p}_\alpha\right) = \tag{2.33}$$

$$= \delta\boldsymbol{\varphi} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_\alpha \boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha\right) = 0. \tag{2.34}$$

Since $\delta\boldsymbol{\varphi}$ is arbitrary, it follows that the vector[6]

$$\boldsymbol{M} \equiv \sum_\alpha \boldsymbol{r}_\alpha \times \boldsymbol{p}_\alpha, \tag{2.35}$$

called the *angular momentum* or *moment of momentum* of the system, is conserved in the motion of the closed system as a consequence of the space isotropy. Like linear momentum, it is additive, whether or not the particles in the system interact.

2.2.4 Isotropy of time

Assuming the previous findings, the question of a conservation law deriving from time isotropy may arise. Unfortunately, we are not able to speak about consequences of time isotropy because of Euclidean metric, which does not depend on time. Therefore, it is not possible to „rotate time“.[7]

[6] Since the definition involves a radius vector of particles, its values depend in general on the choice of origin and its changes. More about that is available in [Nov10].

[7] This is possible in i.e. General relativity

This chapter has been written up by summarizing the author's Bachelor thesis [Nov10], which was based on the introduction to Lagrangian mechanics written by Landau and Lifshitz [LL76].

Chapter 3

Classical continuum mechanics

In this chapter, the basic results on the field of laws of conservation are recapitulated, which are, in the context of continuum mechanics, called *balance laws* or simply *balances*. The reason to do so is to ingeminate and emphasize the techniques that are common for Classical continuum mechanics. By the adjective *Classical* is meant that for representation of considered quantities the field description (theory) will be used.

Let us introduce the general principles which are used to derive this balances. At first we shall start with the *General balance law*.

3.1 General balance law

Let us derive the General balance law for some general extensive quantity $\mathbf{\Phi}(\mathrm{t})$, which characterizes evolution of a system. The system at time $t = 0$ occupies a region of volume $\mathcal{V}(0) \equiv \mathcal{V}_0$, which is called *reference configuration*. Points $\boldsymbol{X}$ that belong to reference configuration are called *material* points.

As have been said, the system is evolving, so the region which the system occupies is changing during this evolution. At the time $t \in \mathbb{R}^+$ the system occupies a region of volume $\mathcal{V}(\mathrm{t}) \equiv \mathcal{V}$. This region is analogously called *current configuration*.

As it is mentioned in the origin of this chapter, in classical theory quantities are considered as fields. Together with the requirement to describe the motion of continuum in Classical continuum mechanics it is usual to use Eulerian approach, which is more convenient in the depicted situation. Therefore, the Eulerian approach will also be used in the thesis[1], and this fact will not be emphasized later in this chapter.

Density of quantity

If we consider that density $\varphi(\boldsymbol{x}, t)$ of the general extensive quantity $\mathbf{\Phi}(\mathrm{t})$ is defined by relation

$$\varphi(\boldsymbol{x}, t) \equiv \lim_{\Delta\mathcal{V} \to 0} \frac{\Delta\mathbf{\Phi}(\mathrm{t})}{\Delta\mathcal{V}}, \tag{3.1}$$

we get the relation between them

$$\mathbf{\Phi}(\mathrm{t}) = \int_{\mathcal{V}} \varphi(\boldsymbol{x}, \mathrm{t}) \, \mathrm{d}\nu. \tag{3.2}$$

[1] Lagrangian approach is discussed in [Nov10] or more generally in [Mar99]. In both books the discontinuities of surfaces are further considered. However, for the purposes of the thesis, they are not significant.

Formulation of general balance law

It is well known and empirically confirmed that total time change of the quantity $\mathbf{\Phi}(\mathrm{t})$ is caused by flow of the quantity through the boundary or by generation or extinction of the quantity $\mathbf{\Phi}(\mathrm{t})$ inside the body (system). This fact can be written

$$\frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{d}t} = \dot{\overline{\mathbf{\Phi}}} = \mathcal{J}(\mathbf{\Phi}) + \mathcal{P}(\mathbf{\Phi}), \tag{3.3}$$

where $\mathcal{J}(\mathbf{\Phi})$ is *total flux* of quantity $\mathbf{\Phi}$ through the body surface and $\mathcal{P}(\mathbf{\Phi})$ is *total production* (generation) of quantity $\mathbf{\Phi}$ in the whole body.

Let us define the total flux

$$\mathcal{J}(\mathbf{\Phi}) = \int_{\partial\mathcal{V}} \boldsymbol{j}(\mathbf{\Phi}) \cdot \mathrm{d}\boldsymbol{a} \tag{3.4}$$

and the total production

$$\mathcal{P}(\mathbf{\Phi}) = \int_{\mathcal{V}} \sigma(\mathbf{\Phi})\, \mathrm{d}\nu, \tag{3.5}$$

where $\boldsymbol{j}$ or $\sigma(\mathbf{\Phi})$ is density of the total flux or total production.

If we take all these circumstances into account, and substitute it within (3.3), we get

$$\dot{\overline{\int_{\mathcal{V}} \varphi(\boldsymbol{x}, \mathrm{t})\, \mathrm{d}\nu}} = \int_{\partial\mathcal{V}} \boldsymbol{j}(\mathbf{\Phi}) \cdot \mathrm{d}\boldsymbol{a} + \int_{\mathcal{V}} \sigma(\mathbf{\Phi})\, \mathrm{d}\nu. \tag{3.6}$$

Now, by using Transport theorem for material derivative and by Green theorem

$$\int_{\mathcal{V}} \Big[[\varphi(\boldsymbol{x}, \mathrm{t})]_{,\mathrm{t}} + \nabla \cdot (\varphi(\boldsymbol{x}, \mathrm{t})\boldsymbol{v}) \Big]\ \mathrm{d}\nu = \int_{\mathcal{V}} [\nabla \cdot (\boldsymbol{j}(\mathbf{\Phi})) + \sigma(\mathbf{\Phi})]\ \mathrm{d}\nu \tag{3.7}$$

is obtained[2].

Since the choice of control volume $\mathcal{V}$ is arbitrary, the balance (3.7) will be fulfilled only if the integrand identically equals zero, so

$$[\varphi(\boldsymbol{x}, \mathrm{t})]_{,\mathrm{t}} + \nabla \cdot (\varphi(\boldsymbol{x}, \mathrm{t})\boldsymbol{v} - \boldsymbol{j}(\mathbf{\Phi})) - \sigma(\mathbf{\Phi}) = 0 \quad \text{for} \quad \forall(\boldsymbol{x}, \mathrm{t}) \in \mathcal{V} \times \mathbb{R}^+, \tag{3.8}$$

where the last result is the *General balance law* we have been searching for.

3.2 Balance laws

Let us introduce the same basic balance laws of the closed system, which are studied in the thesis. At this moment the quest is quite simple, because the Balance law for quantity $\mathbf{\Phi}$, as has been shown in the previous section, is derived from the relation 3.8 just by appropriate choice of its ordered triple $[\varphi; \boldsymbol{j}; \sigma]$.

[2]Rich mathematical and physical background of this „pass“ is available in Maršík's book [Mar99].

3.2.1 Balance of mass

This balance follows from the fact that in a closed system the total amount of mass (weight) does *not* change. Thus

$$\dot{\overline{m}} = 0.$$

The ordered triple is $[\rho; 0; 0]$, where ρ is *density*[3] of the mass and is defined analogously to (3.2).

From the General balance law (3.8) we get the Balance of mass in form

$$[\rho]_{,\mathrm{t}} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{3.9}$$

This relation is usually called *continuity equation*.

3.2.2 Balance of linear momentum

In this case, as a consequence of the conservation law of linear momentum (2.23), we take the ordered triple $[\rho\boldsymbol{v}; \mathfrak{t}; \rho\boldsymbol{f}]$, where $\boldsymbol{v}$ is velocity in Eulerian sense, $\mathfrak{t}$ is Cauchy stress tensor and $\boldsymbol{f}$ is a density of body force, which is acting on the body.

From the relation (3.8) we get

$$[\rho\boldsymbol{v}]_{,\mathrm{t}} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} - \mathfrak{t}) - \rho \boldsymbol{f} = 0, \tag{3.10}$$

which is the *Balance of linear momentum* we have been looking for.

3.2.3 Balance of angular momentum

This balance is also a result of its conservation law. The density of angular momentum is defined as an outer product of radius vector $\boldsymbol{r}$ and linear momentum $\rho\boldsymbol{v}$

$$\boldsymbol{r} \wedge \rho\boldsymbol{v} = \rho\left(\boldsymbol{r} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{r}\right), \tag{3.11}$$

where the radius vector $\boldsymbol{r}$ is defined $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}_0$, where $\boldsymbol{y}_0$ is an origin of a system of coordinates in which the system is described. The outer product is used due to its better properties accordingly to the change of the system of coordinates. The outer product is transformed as *antisymmetric* tensor in contrast to axial vector like transformation of angular momentum, which is defined as vector product, e.g. in Brdička [BSS05].

The density of surface forces is analogously defined as an outer product of radius vector and stress vector $\boldsymbol{\tau}$, which is defined as a projection of stress tensor $\mathfrak{t}$ into a direction of outer normal $\boldsymbol{n}$ of the surface of the body

$$\boldsymbol{\tau} = \mathfrak{t}^T \boldsymbol{n} = \tau^j = t^{ij} n_i.$$

So the ordered triple is $[\boldsymbol{r} \wedge \rho\boldsymbol{v}; \boldsymbol{r} \wedge \boldsymbol{\tau}; \boldsymbol{r} \wedge \rho\boldsymbol{f}]$ and then from (3.8) we get

$$[\boldsymbol{r} \wedge \rho\boldsymbol{v}]_{,\mathrm{t}} + \nabla_k \cdot \big(\rho(\boldsymbol{r} \wedge \rho\boldsymbol{v})v^k - (\boldsymbol{r} \wedge \boldsymbol{\tau})\big) - \boldsymbol{r} \wedge \rho\boldsymbol{f} = 0. \tag{3.12}$$

By using the definition of outer product in the last relation, an outer product of $\boldsymbol{r}$, the balance of linear momentum (3.10) and some formal adapting, implies that

$$\mathfrak{t} = \mathfrak{t}^T \tag{3.13}$$

which means the symmetry of the stress tensor tensor[4].

[3] For simplicity we call it just density.

[4] in the case of nonpolar material

3.2.4 Balance of mechanical energy

The balance of mechanical energy is a simple consequence of the linear momentum balance. Using a scalar product of the relation (3.10) and velocity $\boldsymbol{v}$, together with Leibnitz rule and continuity equation (3.9), gives us

$$\left[\rho\frac{|\boldsymbol{v}|^2}{2}\right]_{,\mathrm{t}} + \nabla_i \cdot \left(\frac{1}{2}\rho\,|\boldsymbol{v}|^2\, v^i - (\mathsf{t}\boldsymbol{v})_i\right) + \mathsf{t}\cdot\nabla^T(\boldsymbol{v}) - \rho\boldsymbol{f}\cdot\boldsymbol{v} = 0, \qquad (3.14)$$

which is the balance of mechanical energy. It should be noted that by applying the relation (3.14) the density of mechanical energy has a form $\rho\frac{|\boldsymbol{v}|^2}{2}$.

3.2.5 Balance of total energy

In distinction from the balance of mechanical energy, in balance of total energy it is needed to reflect not just the density of mechanical energy, but the internal energy u, heat flux $\boldsymbol{q}$ and an energy source[5] $\tilde{q}$ as well. The ordered triple has form $\left[\rho(\frac{|\boldsymbol{v}|^2}{2}+u); \mathsf{t}\boldsymbol{v}-\boldsymbol{q}; \rho\boldsymbol{f}\cdot\boldsymbol{v}+\tilde{q}\right]$. That implies that the balance of total energy has a form

$$\left[\rho\left(\frac{|\boldsymbol{v}|^2}{2}+u\right)\right]_{,\mathrm{t}} + \nabla_i \cdot \left(\rho\left(\frac{|\boldsymbol{v}|^2}{2}+u\right)v^i - (\mathsf{t}\boldsymbol{v})_i + q_i\right) - \rho\boldsymbol{f}\cdot\boldsymbol{v} - \tilde{q} = 0. \quad (3.15)$$

One of the consequences of the total energy balance is a *balance of internal energy*[6]

$$[\rho u]_{,\mathrm{t}} + \nabla\cdot(\rho u\boldsymbol{v} + \boldsymbol{q}) - \mathsf{t}\cdot\nabla^T(\boldsymbol{v}) - \tilde{q} = 0. \qquad (3.16)$$

The balance of internal energy is continuum mechanics equivalent of the *First law of thermodynamics*.

Specific enthalpy balance

Let us recall the specific enthalpy that is defined as

$$h = u + \frac{p}{\rho}.$$

Before we start, it is convenient to segregate the force provided by the surface forces. The stress tensor can be separated into two parts

$$\mathsf{t} = \mathsf{t}^{\mathrm{el}} + \mathsf{t}^{\mathrm{dis}}, \qquad (3.17)$$

where t^{el} is an elastic part and $\mathsf{t}^{\mathrm{dis}}$ is a dissipative part. The stress tensor can also be separated to the volumetric and deviatoric parts

$$\mathsf{t} = \frac{1}{3}\mathrm{tr}(\mathsf{t})\,\mathbb{I} + \bar{\mathsf{t}}^{dev}$$

[5] In this general sense, there is no need to specify it.

[6] This relation is obtained just by subtracting the balance of mechanical energy form the balance of total energy

and in the case of

$$\mathrm{tr}\left(\mathrm{t}^{\mathrm{dis}}\right) = 0,$$

we are allowed to write

$$\frac{1}{3}\mathrm{tr}\left(\mathrm{t}\right) = \mathrm{t}^{\mathrm{el}} = -p \qquad \overline{\mathrm{t}}^{dev} = \mathrm{t}^{\mathrm{dis}}. \tag{3.18}$$

It is seen that the volumetric forces are managed through the pressure forces and the dissipative forces through the viscous friction. When we rewrite the balance of internal energy (3.16) in terms of p and $\mathrm{t}^{\mathrm{dis}}$, we get

$$\rho\dot{\overline{u}} = -p\nabla\cdot(\boldsymbol{v}) + \mathrm{t}^{\mathrm{dis}}\cdot\nabla^T(\boldsymbol{v}) - \nabla\cdot(\boldsymbol{q}),$$

where the heat production is neglected. Next, we eliminate the constituent $\nabla\cdot(\boldsymbol{v})$, which corresponds to relative expansion rate, by using the continuity equation in shape

$$\dot{\overline{\rho}} + \rho\nabla\cdot(\boldsymbol{v}) = 0,$$

by which we get

$$\rho\dot{\overline{u}} = -\frac{p}{\rho}\dot{\overline{\rho}} + \mathrm{t}^{\mathrm{dis}}\cdot\nabla^T(\boldsymbol{v}) - \nabla\cdot(\boldsymbol{q}),$$

which can be rearranged to the form

$$\rho\left(\dot{\overline{u}} - \frac{p}{\rho^2}\dot{\overline{\rho}}\right) = \mathrm{t}^{\mathrm{dis}}\cdot\nabla^T(\boldsymbol{v}) - \nabla\cdot(\boldsymbol{q}).$$

Now, from the definition of specific enthalpy h, we have

$$\dot{\overline{h}} = \dot{\overline{u}} - \frac{p}{\rho^2}\dot{\overline{\rho}} + \rho^{-1}\dot{\overline{p}}.$$

So, the energy balance in term of *specific* enthalpy becomes

$$\rho\dot{\overline{h}} = \dot{\overline{p}} + \mathrm{t}^{\mathrm{dis}}\cdot\nabla^T(\boldsymbol{v}) - \nabla\cdot(\boldsymbol{q}). \tag{3.19}$$

Total enthalpy balance

Let us derive the total enthalpy h_T balance, first by recalling the definition of total enthalpy

$$h_T = h + \frac{|\boldsymbol{v}|^2}{2} + \Phi,$$

where Φ is a potential of the field of body forces $\boldsymbol{f}$

$$\boldsymbol{f} = -\nabla(\Phi).$$

Applying material derivative on the last relation, we get

$$\dot{\overline{h}} = \dot{\overline{h_T}} - \frac{\dot{\overline{|\boldsymbol{v}|^2}}}{2} - \dot{\overline{\Phi}}.$$

Since the potential Φ is not an explicit function of time, the material derivative is

$$\dot{\overline{\Phi}} = \left(\left(\frac{\partial\cdot}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial\cdot}{\partial\boldsymbol{x}}\right)_{\mathrm{t}}\left(\frac{\partial\boldsymbol{x}}{\partial t}\right)_{\boldsymbol{X}}\right)\Phi() = (\boldsymbol{v}\cdot\nabla)(\Phi)$$

and the material derivative of the kinematic part is

$$\frac{\dot{\overline{v_i v_i}}}{2} = v_i \dot{\overline{v_i}}$$

due the obvious symmetry.

Next, we derive the total entropy balance

$$\begin{aligned}
\rho \dot{\overline{h}}_T &= \rho \dot{\overline{h}} - \rho v_i \dot{\overline{v_i}} - \rho v_j \nabla\left(\Phi\right) \overset{(3.19)}{=} \quad \overset{i \leftrightarrow j}{} \\
&= \dot{\overline{p}} + t_{ij}^{\text{dis}} \cdot \nabla_i\left(v_j\right) - \nabla \cdot \left(\boldsymbol{q}\right) - \rho v_j \dot{\overline{v_j}} - \rho v_j \underbrace{\nabla\left(\Phi\right)}_{\boldsymbol{f}} = \\
&= \left(\frac{\partial p}{\partial t}\right) + v_j \underline{\left(-\rho \dot{\overline{v_j}} + \nabla\left(p\right) - \nabla_i \cdot \left(t_{ij}^{\text{dis}}\right) + \boldsymbol{f}\right)} - \nabla \cdot \left(\boldsymbol{q}\right) + \nabla_i \cdot \left(t_{ij}^{\text{dis}} v_j\right).
\end{aligned}$$

The linear momentum balance (3.10) implies that the underlined constituent equals zero. So the energy balance in term of *total* enthalpy becomes

$$\rho \dot{\overline{h}}_T = \left(\frac{\partial p}{\partial t}\right) - \nabla \cdot \left(\boldsymbol{q}\right) + \nabla_i \cdot \left(t_{ij}^{\text{dis}} v_j\right). \tag{3.20}$$

Heat transfer balance

As the denomination implies, this balance describes the conduction (propagation) of heat through a material.

As a starting point for this balance, the internal energy balance

$$\rho\left(\dot{\overline{u}} - \frac{p}{\rho^2}\dot{\overline{\rho}}\right) = \mathsf{t}^{\text{dis}} \cdot \nabla^T\left(\boldsymbol{v}\right) - \nabla \cdot \left(\boldsymbol{q}\right)$$

can be taken. Accordingly to the Second law of thermodynamics and the Fourier's law for conduction of heat flux

$$\boldsymbol{q} = -k\nabla\left(T\right), \tag{3.21}$$

where k is the thermal conductivity, we get

$$\rho T \dot{\overline{s}} = \mathsf{t}^{\text{dis}} \cdot \nabla^T\left(\boldsymbol{v}\right) + \nabla \cdot \left(k \nabla\left(T\right)\right). \tag{3.22}$$

In the context of heat conduction, the first term is the energy dissipated into heat by viscosity, and the second is heat conducted into concerned volume. Due to determining the relation

$$s = s(\mathrm{T}, \ldots),$$

we obtain the heat heat transfer balance.

Incompressible Heat transfer balance

In this case, we have

$$T\dot{\overline{s}} = c_p \dot{\overline{T}},$$

where c_p is the specific heat at a constant pressure, but the p is not, strictly speaking, the pressure in the usual sense - the normal force on surface element, but it is a term proportional to the $\nabla \cdot (\boldsymbol{v})$ and by constant a pressure in fluids dynamics is mentioned that

$$\nabla \cdot \left(\boldsymbol{v}\right) = 0. \tag{3.23}$$

So, we have the balance

$$\rho c_p \dot{\overline{T}} = \mathsf{t}^{\text{dis}} \cdot \nabla^T\left(\boldsymbol{v}\right) + \nabla \cdot \left(k \nabla\left(T\right)\right). \tag{3.24}$$

3.2.6 Entropy balance

To derive a relation that would describe the balance of entropy, a slightly different approach will be applied, due to the most possible simplicity and clearness.

At first it should be noted that such an entropy is considered which does not characterize just macroscopic state of the system, but also processes which take place in the considered system[7].

The entropy complies with the *Clausius inequality*

$$T^{-1}\,\mathrm{d}Q \leq \,\mathrm{d}S, \tag{3.25}$$

where $\mathrm{d}Q$ is change of heat and T is temperature.

Assume that each material point satisfies the Clausius inequality (3.25) and that $\mathrm{d}Q$ is a whole amount of heat which the material point exchanges with its vicinity at the temperature T. When the definition of entropy is used in form

$$S = \int_{\mathcal{V}} \rho(\boldsymbol{x},t)s(\boldsymbol{x},t)\,\mathrm{d}\nu, \tag{3.26}$$

with specific entropy s, which is related to unit of volume, we get the *generalized Clausius inequality*

$$\dot{\overline{\int_{\mathcal{V}} \rho s\,\mathrm{d}\nu}} \geq -\int_{\partial\mathcal{V}} T^{-1}\boldsymbol{q}\,\mathrm{d}\boldsymbol{a} + \int_{\mathcal{V}} T^{-1}\tilde{q}\,\mathrm{d}\nu. \tag{3.27}$$

The left hand side of the last relation represents an infinitesimal increment of entropy

$$\mathrm{d}s = \dot{\overline{s}}.$$

Then a total production of entropy is defined

$$\sigma(S) = \rho\dot{\overline{s}} + \nabla\cdot\left(T^{-1}\boldsymbol{q}\right) - T^{-1}\tilde{q} \geq 0, \tag{3.28}$$

from the reason of fulfilling (3.27). The last relation represents *entropy balance law* and is one of the *Clausius - Duhem* balances[8].

At the end, it should be told, that the last relation is frequently written in the form

$$\sigma(S) = \rho\left(\dot{\overline{s}} - T^{-1}\dot{\overline{u}}\right) + (\boldsymbol{q}\cdot\nabla)\left(T^{-1}\right) + T^{-1}\mathfrak{t}\cdot\nabla^T(\boldsymbol{v}) \geq 0, \tag{3.29}$$

which is earned by using the balance of total internal energy (3.16).

The last inequality can be rewritten by the definition of density of dissipation energy π

$$\pi \equiv T\sigma(S) = \rho\left(T\dot{\overline{s}} - \dot{\overline{u}}\right) + T^{-1}(\boldsymbol{q}\cdot\nabla)(T) + \mathfrak{t}\cdot\nabla^T(\boldsymbol{v}) \geq 0. \tag{3.30}$$

The dissipation energy density has dimension of energy and the balance has a fundamental significance in *Non-equilibrium thermodynamics*. The fact of positivity of dissipation energy density represents the most generalized formulation of the Second law of thermodynamics of a one-component system (body).

[7] A deeper commentary is placed e.q. in chapters 5.1. a 5.2. in Maršík [Mar99]

[8] in this case the spacial one

3.3 Material point concept

The next chapters will operate with the concept of *material points*, denoted as $\boldsymbol{X}$. We will study, under some given conditions, the consequences of motion of this point through the continuum. At this time, let us derive the forces that are acting on the material point in a current configuration, when just motion of the material point is taken under consideration. It might seem that on the material point just the classical volume force is acting. However, as it will be later shown, it is *not* that explicit.

Let us start from the linear momentum balance (3.10) with the body force in the shape of potential field $\boldsymbol{f} = -\nabla(\boldsymbol{\phi})$

$$[\rho \boldsymbol{v}]_{,\mathrm{t}} + \nabla_i \cdot (\rho \boldsymbol{v} v_i) - \nabla \cdot (\mathfrak{t}) + \rho \nabla(\phi) = 0. \tag{3.31}$$

Using the Leibnitz rule and the continuity equation (3.9), it could be rewritten to the form

$$\rho \dot{\boldsymbol{v}} = \nabla \cdot (\mathfrak{t}) - \rho \nabla(\phi). \tag{3.32}$$

Due to tensorial identity

$$\boldsymbol{v} \times \nabla \times \boldsymbol{v} = \frac{1}{2}\nabla\left(|\boldsymbol{v}|^2\right) - (\boldsymbol{v} \cdot \nabla)(\boldsymbol{v}), \tag{3.33}$$

the last relation could be rewritten into a form

$$\rho\left([\boldsymbol{v}]_{,\mathrm{t}} + \frac{1}{2}\nabla\left(|\boldsymbol{v}|^2\right) - \boldsymbol{v} \times \nabla \times \boldsymbol{v}\right) = \nabla \cdot (\mathfrak{t}) - \rho \nabla(\phi).$$

When we define *total energy* ϵ_0 of the material point by the relation

$$\epsilon_0 = \frac{|\boldsymbol{v}|^2}{2} + \phi$$

and consider that a quantity

$$\boldsymbol{w} \equiv \nabla \times (\boldsymbol{v}),$$

which is called *vorticity*, then we will get the equation

$$[\boldsymbol{v}]_{,\mathrm{t}} + \nabla(\epsilon_0) = \boldsymbol{v} \times \boldsymbol{w} + \rho^{-1}\nabla \cdot (\mathfrak{t}). \tag{3.34}$$

At this point, it should be mentioned that this case could be described also using the Hamilton equations

$$\frac{\partial H}{\partial \boldsymbol{q}} = -\boldsymbol{p} \qquad \frac{\partial H}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}},$$

where H is a *Hamiltonian*. The first of these equations can be reformulated, assuming unit mass, into the form

$$[\boldsymbol{v}]_{,\mathrm{t}} + \nabla(h) = 0,$$

where h is the density of a Hamiltonian, in this case defined as a sum of kinetic and potential energy, therefore

$$h = \frac{1}{2}|\boldsymbol{v}|^2 + \phi = \epsilon_0.$$

Finally, it is seen that the right hand side of the relation (3.34) is the sought force, which is related to the motion of the material point in the material continuum. This force could also be interpreted as a consequence of surface forces that are related to the stress tensor.

3.4 Conclusions of the Classical continuum mechanic on the field of vorticity generation

At has been pointed, the thesis scopes the influence of entropy gradient on the vorticity $\boldsymbol{w}$. Thus, let us derive some conclusions, which provided by Classical continuum mechanics in order to build up some grounds, it will be possible to compare the results of Variational mechanics.

3.4.1 Crocco's theorem

The main assumption of the theorem is the stationarity of processes. This yields to the linear momentum balance

$$\nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \nabla \cdot (\mathfrak{t}) + \rho \boldsymbol{f},$$

where the left hand side can be modified using the continuity equation to the form

$$\nabla_j \cdot (\rho v_i v_j) = v_i \underbrace{\nabla_j \cdot (\rho v_j)}_{0} + \rho v_j \nabla_j (v_i) .$$

Due to identity (3.33)

$$\boldsymbol{v} \times \nabla \times \boldsymbol{v} = \frac{1}{2} \nabla \left(|\boldsymbol{v}|^2 \right) - (\boldsymbol{v} \cdot \nabla) (\boldsymbol{v}) .$$

the balance of linear momentum can be written as

$$-\rho \boldsymbol{v} \times \nabla \times \boldsymbol{v} = -\rho \frac{1}{2} \nabla \left(|\boldsymbol{v}|^2 \right) + \nabla \cdot (\mathfrak{t}) + \rho \boldsymbol{f}.$$

Let us focus to the right hand side, recalling that

$$\mathfrak{t} = \frac{1}{3} \mathrm{tr}\, \mathfrak{t} \mathbb{I} + \bar{\mathfrak{t}}^{dev}$$

$$\frac{1}{3} \mathrm{tr}\, \mathfrak{t} = -p \qquad \bar{\mathfrak{t}}^{dev} = \mathfrak{t}^{\mathrm{dis}},$$

where p is a static pressure and $\mathfrak{t}^{\mathrm{dis}}$ is a dissipative part of the stress tensor. Further the body forces will be defined through the potential of body forces with the relation

$$\boldsymbol{f} = -\nabla (\Phi) .$$

All these assumptions imply the linear momentum balance

$$-\rho \boldsymbol{v} \times \nabla \times \boldsymbol{v} = -\rho \frac{1}{2} \nabla \left(|\boldsymbol{v}|^2 \right) + \nabla \cdot \left(-p \mathbb{I} + \mathfrak{t}^{\mathrm{dis}} \right) - \rho \nabla (\Phi) .$$

As has been said, the stationary case is considered in this derivation. In the stationary processes the *Second law of thermodynamics* can be formulated

$$\nabla (u) = T \nabla (s) - p \nabla \left(\rho^{-1} \right) . \qquad (3.35)$$

When we start from the last relation and do some formal adaptations

$$\nabla\left(u\right)-T\nabla\left(s\right)=-p\nabla\left(\rho^{-1}\right)=-\left(\nabla\left(\frac{p}{\rho}\right)-\frac{\nabla\left(p\right)}{\rho}\right),$$

we obtain the linear momentum balance in the well-known form

$$-\boldsymbol{v}\times\nabla\times\boldsymbol{v}=-\nabla\left(\underbrace{\frac{1}{2}\left|\boldsymbol{v}\right|^{2}+\Phi+u+\frac{p}{\rho}}_{h_T}\right)+T\nabla\left(s\right)+\frac{\nabla\cdot\left(\mathsf{t}^{\mathrm{dis}}\right)}{\rho} \tag{3.36}$$

of the Crocco's theorem. It should be noted that the derivation of the theorem is performed just by considering the *phenomenological* balance laws, as it is seen above. Originally this relation was revealed by Luigi Crocco in the article [Cro37]. The previous process of derivation of this theorem is similar to the procces which was used in the Batchelor's book [Bat00], although there have not been any indications on the requirement of isentropic flow. As Batchelor also pointed, the theorem can be achieved by the stagnation enthalpy approach.

3.4.2 Extended Crocco's theorem

Unfortunately, the previous approach based on the Second law of thermodynamics can not be generalized to non-stationary cases in the exactly same manner. To accomplish the generalization, different assumptions are needed. The starting point will be the same- the general linear momentum balance. As has been shown, the linear momentum balance (3.10) can be rewritten in the form

$$\left[\boldsymbol{v}\right]_{,\mathrm{t}}-\boldsymbol{v}\times\nabla\times\boldsymbol{v}=-\nabla\left(\frac{\left|\boldsymbol{v}\right|^{2}}{2}+\Phi+\frac{p}{\rho}\right)+p\nabla\left(\rho^{-1}\right)+\rho^{-1}\nabla\cdot\left(\mathsf{t}^{\mathrm{dis}}\right).$$

Formally, we are allowed to add and subtract an internal energy u on the right hand side of the previous relation. It implies for the right hand side

$$-\nabla\left(\underbrace{\frac{\boldsymbol{v}^{2}}{2}+\Phi+\frac{p}{\rho}+u}_{h_T}\right)+\nabla\left(u\right)+p\nabla\left(\rho^{-1}\right)+\rho^{-1}\nabla\cdot\left(\mathsf{t}^{\mathrm{dis}}\right),$$

which looks really close to the result sought. However, at this moment we encounter difficulties, as it seems that the Second law of thermodynamics does not persist in the shape of the equation (3.35). In the case of non-stationary continuum mechanics, the differentials become the material derivations and the Second law of thermodynamics[9] for *Thermo-viscous fluids* is written

$$\dot{u}=T\dot{s}-p\overline{\rho^{-1}}^{\cdot}. \tag{3.38}$$

[9] The permutation of this relation, called *Gibbs definition of entropy*, is often used in the form

$$\dot{s}=^{-1}T\dot{u}+\frac{p}{T}\overline{\rho^{-1}}^{\cdot}. \tag{3.37}$$

It should be noted that all balances interpreting Second law of thermodynamics remain true as equalities only in the case of *locally* equilibrium processes. Under conditions of non-equilibrium processes, they become inequalities.

The last relation can be modified

$$\dot{\overline{u}} = T\dot{\overline{s}} + \frac{p}{\rho^2}\dot{\overline{\rho}}. \tag{3.39}$$

And these conclusions give us a hint to achieve the goal.

Strictly from the mathematical point of view, by the assumption that the internal energy u is a function of the entropy s and the density ρ,

$$u = u(\mathrm{s}, \rho),$$

we will be able to derive the sought theorem. In the function representation of internal energy both variables depend on location $\boldsymbol{x}$ and time t.

We will scope the internal energy in the Eulerian sense, as the mathematical abstraction, i.e. field quantity, which is not necessarily connected with a tangible particle. Basically, there is no assurance that the particle occupies the position $\boldsymbol{x}$ at the time t.

The location in the Eulerian sense is a function of the location in reference configuration $\boldsymbol{X}$ and *implicit* time τ. The reason to distinguish between t and τ is to highlight the deeper dependence on implicit time through the Eulerian location. These differences will be illustrated on the definition[10] of the material derivation of scalar[11] $\varphi = \varphi(\boldsymbol{x}(\boldsymbol{X}, \tau), t)$. In reference configuration the material derivation is defined as a partial time derivation of the scalar for the given material point $\boldsymbol{X}$ (i.e. Lagrangian location)

$$\dot{\overline{\varphi}} = \dot{\overline{\varphi}}(\boldsymbol{X}, t) = \left.\frac{\partial \varphi(\boldsymbol{X}, t)}{\partial t}\right|_{\boldsymbol{X}}.$$

This fact is taken as a starting point of deriving the appropriate generalization on the case of current configuration. By the chain rule,

$$\begin{aligned}
\dot{\overline{\varphi}} = \dot{\overline{\varphi}}(\boldsymbol{x}(\boldsymbol{X}, \tau), t) &= \left.\frac{\partial \varphi(\boldsymbol{x}(\boldsymbol{X}, \tau), t)}{\partial t}\right|_{\boldsymbol{X}=\boldsymbol{X}(\boldsymbol{x}, \tau)} = \\
&= \left(\frac{\partial \varphi}{\partial \boldsymbol{x}}\right)_{\mathrm{t} \cap \boldsymbol{X}} \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{X}} \left(\frac{\partial \tau}{\partial t}\right)_{\boldsymbol{X}} + \left(\frac{\partial \varphi}{\partial t}\right)_{\boldsymbol{x} \cap \boldsymbol{X}} \stackrel{t=\tau}{=} \\
&= \left(\frac{\partial \varphi}{\partial \boldsymbol{x}}\right)_{\mathrm{t}} \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{X}} + \left(\frac{\partial \varphi}{\partial t}\right)_{\boldsymbol{x}},
\end{aligned} \tag{3.40}$$

where the fact that φ does not depend explicitly on $\boldsymbol{X}$ is used. Let us derive a relation for the material derivative of internal energy

$$u = u(\mathrm{s}(\boldsymbol{x}(\boldsymbol{X}, \tau), \mathrm{t}), \rho(\boldsymbol{x}(\boldsymbol{X}, \tau), \mathrm{t})). \tag{3.41}$$

[10] We switch to the notation $\frac{\partial}{\partial}$ from the more illustrative reason.

[11] A richer background on the field of material derivatives is placed e.g. in the Author's Bachelor thesis [Nov10] or in Maršík [Mar99].

By starting with the individual components of the material derivative,

$$\begin{aligned}
\left(\frac{\partial u}{\partial t}\right)_{\boldsymbol{x}} &= \left(\frac{\partial u(\mathrm{s}(\boldsymbol{x},\mathrm{t}),\rho(\boldsymbol{x},\mathrm{t}))}{\partial t}\right)_{\boldsymbol{x}} = \\
&= \left(\frac{\partial u}{\partial s}\right)_{\rho\,\cap\,\boldsymbol{x}}\left(\frac{\partial s}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial u}{\partial \rho}\right)_{\mathrm{s}\,\cap\,\boldsymbol{x}}\left(\frac{\partial \rho}{\partial t}\right)_{\boldsymbol{x}} = \\
&= \left(\frac{\partial u}{\partial s}\right)_{\rho}\left(\frac{\partial s}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial u}{\partial \rho}\right)_{\mathrm{s}}\left(\frac{\partial \rho}{\partial t}\right)_{\boldsymbol{x}} \qquad (3.42)\\
\left(\frac{\partial u}{\partial \boldsymbol{x}}\right)_{\mathrm{t}} &= \left(\frac{\partial u(\mathrm{s}(\boldsymbol{x},\mathrm{t}),\rho(\boldsymbol{x},\mathrm{t}))}{\partial \boldsymbol{x}}\right)_{\mathrm{t}} = \\
&= \left(\frac{\partial u}{\partial s}\right)_{\rho\,\cap\,\mathrm{t}}\left(\frac{\partial s}{\partial \boldsymbol{x}}\right)_{\mathrm{t}} + \left(\frac{\partial u}{\partial \rho}\right)_{\mathrm{s}\,\cap\,\mathrm{t}}\left(\frac{\partial \rho}{\partial \boldsymbol{x}}\right)_{\mathrm{t}} = \\
&= \left(\frac{\partial u}{\partial s}\right)_{\rho}\left(\frac{\partial s}{\partial \boldsymbol{x}}\right)_{\mathrm{t}} + \left(\frac{\partial u}{\partial \rho}\right)_{\mathrm{s}}\left(\frac{\partial \rho}{\partial \boldsymbol{x}}\right)_{\mathrm{t}}, \qquad (3.43)
\end{aligned}$$

it is seen that

$$\dot{u} = \left(\frac{\partial u}{\partial s}\right)_{\rho}\left(\frac{\partial s}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial u}{\partial \rho}\right)_{\mathrm{s}}\left(\frac{\partial \rho}{\partial t}\right)_{\boldsymbol{x}} + \qquad (3.44)$$

$$+\left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{X}}\left(\left(\frac{\partial u}{\partial s}\right)_{\rho}\left(\frac{\partial s}{\partial \boldsymbol{x}}\right)_{\mathrm{t}} + \left(\frac{\partial u}{\partial \rho}\right)_{\mathrm{s}}\left(\frac{\partial \rho}{\partial \boldsymbol{x}}\right)_{\mathrm{t}}\right) = \qquad (3.45)$$

$$= \left(\frac{\partial u}{\partial s}\right)_{\rho}\dot{s} + \left(\frac{\partial u}{\partial \rho}\right)_{\mathrm{s}}\dot{\rho}. \qquad (3.46)$$

By comparison with the Second law of Thermodynamics (3.39) it is obtained that

$$\left(\frac{\partial u}{\partial s}\right)_{\rho} = T \qquad \left(\frac{\partial u}{\partial \rho}\right)_{\mathrm{s}} = \frac{p}{\rho^2}. \qquad (3.47)$$

Using this conclusions it is possible to rewrite the right hand side of the linear momentum balance to the form

$$\begin{aligned}
&-\nabla\left(h_T\right) + \nabla\left(u\right) + p\nabla\left(\rho^{-1}\right) + \rho^{-1}\nabla\cdot\left(\mathfrak{t}^{\mathrm{dis}}\right) = \\
&= -\nabla\left(h_T\right) + \left(\frac{\partial u}{\partial s}\right)_{\rho}\nabla\left(s\right) + \left(\frac{\partial u}{\partial \rho}\right)_{\mathrm{s}}\nabla\left(\rho\right) + p\nabla\left(\rho^{-1}\right) + \rho^{-1}\nabla\cdot\left(\mathfrak{t}^{\mathrm{dis}}\right) = \\
&= -\nabla\left(h_T\right) + T\nabla\left(s\right) + \rho^{-1}\nabla\cdot\left(\mathfrak{t}^{\mathrm{dis}}\right),
\end{aligned}$$

which we have been looking for. Altogether, it implies the linear momentum balance in the shape

$$\left[\boldsymbol{v}\right]_{,\mathrm{t}} - \boldsymbol{v}\times\nabla\times\left(\boldsymbol{v}\right) = -\nabla\left(h_T\right) + T\nabla\left(s\right) + \rho^{-1}\nabla\cdot\left(\mathfrak{t}^{\mathrm{dis}}\right). \qquad (3.48)$$

3.5 Balance of linear momentum for solids

Further, let us introduce the linear momentum balance for solids, the analogy to the Extended Crocco's theorem with Thermo-viscous-elastic material instead of Thermo-viscous fluid. Since it is the an analogy, the approach to derive this kind of balance will be analogical, too. The main difference is obvious in the assumption of the internal energy dependence. In the case of Thermo-viscous-elastic materials the specific entropy is defined by

$$\dot{\bar{s}} = T^{-1}\dot{\bar{u}} - T^{-1}\rho^{-1}\mathfrak{t}^{\mathrm{el}} : \dot{\bar{\mathfrak{e}}}, \tag{3.49}$$

which is derived from the Second law of thermodynamics. The Second law for Thermo-viscous-elastic material has the shape

$$\dot{\bar{u}} = T\dot{\bar{s}} + \rho^{-1}\mathfrak{t}^{\mathrm{el}} : \dot{\bar{\mathfrak{e}}}. \tag{3.50}$$

Therefore, when the internal energy is taken as a function of specific entropy s and deformation tensor $\mathfrak{e}$, we will get

$$\left(\frac{\partial u}{\partial s}\right)_{\mathfrak{e}} = T \qquad \left(\frac{\partial u}{\partial \mathfrak{e}}\right)_{\mathrm{s}} = \frac{\mathfrak{t}^{\mathrm{el}}}{\rho}, \tag{3.51}$$

assuming the same conditions like in the case of fluids.

Next, when we start from the linear momentum balance (3.10), by the methods used to derive the Extended Crocco's theorem,

$$\begin{aligned}
\rho\dot{\bar{\boldsymbol{v}}} &= \nabla\cdot(\mathfrak{t}) + \rho\boldsymbol{f}\\
\rho\left([\boldsymbol{v}]_{,\mathrm{t}} + (\boldsymbol{v}\cdot\nabla)(\boldsymbol{v})\right) &= \nabla\cdot(\mathfrak{t}) - \rho\nabla(\Phi)\\
\rho\left([\boldsymbol{v}]_{,\mathrm{t}} - \boldsymbol{v}\times\nabla\times(\boldsymbol{v}) + \nabla\left(\frac{|\boldsymbol{v}|^2}{2}\right)\right) &= \nabla\cdot(\mathfrak{t}) - \rho\nabla(\Phi) \pm \rho\nabla(u)\\
\rho\left([\boldsymbol{v}]_{,\mathrm{t}} - \boldsymbol{v}\times\nabla\times(\boldsymbol{v})\right) &= -\rho\nabla\left(\frac{|\boldsymbol{v}|^2}{2} + u + \Phi\right) + \nabla\cdot(\mathfrak{t}) + \rho\nabla(u),
\end{aligned}$$

by the definition of the *total energy* of the Thermo-viscous-elastic material

$$\epsilon_T = \frac{|\boldsymbol{v}|^2}{2} + \Phi + u \tag{3.52}$$

and by using the differentials, we obtain the linear momentum balance in term of total energy

$$[\boldsymbol{v}]_{,\mathrm{t}} - \boldsymbol{v}\times\nabla\times(\boldsymbol{v}) = -\nabla(\epsilon_T) + T\nabla(s) + \rho^{-1}\nabla\cdot(\mathfrak{t}) + \rho^{-1}{t^{\mathrm{el}}}_{kl}\nabla(e_{kl}), \tag{3.53}$$

which we have been searching for.

Chapter 4

Variational principles in continuum mechanics

Motivation

One of the motivations of scoping the topic for continuum mechanics from the *Variational principles* point of view is to study the consequences of the possibility to divide the Second law of thermodynamics into two parts, how is it made in the Classical continuum mechanics. In particular, the assumptions which have been made in the Extended Crocco's theorem derivation, imply that the Second law of thermodynamics

$$\mathrm{d}u = T\,\mathrm{d}s - p\,\mathrm{d}\rho^{-1}$$

can be, in the Classical continuum mechanics, split into two parts, where one

$$[u]_{,\mathrm{t}} = T\,[s]_{,\mathrm{t}} - p\,\left[\rho^{-1}\right]_{,\mathrm{t}}$$

describes the local changes of given quantities and the second

$$(\boldsymbol{v}\cdot\nabla)\,(u) = T\,(\boldsymbol{v}\cdot\nabla)\,(s) - p\,(\boldsymbol{v}\cdot\nabla)\,\left(\rho^{-1}\right)$$

corresponds with the convective changes.

This is undoubtedly true in the stationary limit, where the first of the splitted equations is identically zero. This fact has been used for the derivation of the Crocco's theorem. However, in the non-stationary limit, this possibility is at least quite strange.

The difficulties are connected with the material derivative, that represent the differential d from the Second law of thermodynamics in the case of the Classical continuum mechanics.

The advantage of the Variational principles is that from point of view the differential d is "replaced" by variational derivative

$$\frac{\delta\cdot}{\delta\boldsymbol{\chi}},$$

which is, unlike the Classical continuum mechanics, *always* connected with some material point P. But first things first.

Introduction and preliminaries

The Variational principles are the continuum mechanics analogy of the Hamilton's theory. The connection with the Hamilton's theory of discrete particles, which was recalled in the section 2.1, is preserved, when the Lagrangian coordinates are used. In that case, the summation is just replaced by the integration and no other changes are needed. However, if the Eulerian description is adopted, this close similarity is lost. It becomes difficult to highlight the connection between the Variational principles as a mathematical device and the physical background[1]. Nevertheless, the Eulerian description is preferable in the case of fluids and fluids like materials, so this description will be applied. Actually, we will also refer to a motion of *material point* as was introduced in the previous section. The clarification of the notion used follows.

Framework

As has been said in the chapter 2, the action, in the case of a discrete problem, has the shape (2.2)

$$\mathscr{A} = \int_{t_1}^{t_2} L\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right)\, \mathrm{d}t,$$

but at this moment the continuum mechanics are considered, so the Lagrangian L is defined by its own density $\mathcal{L}$ through an integral value over the considered system (body), which, at time t occupies the volume $\mathcal{V} = \mathcal{V}(\mathrm{t})$, so in the Eulerian description

$$L = \int_{\mathcal{V}} \mathcal{L}(\boldsymbol{x}, \mathrm{t})\, \mathrm{d}\nu. \tag{4.1}$$

It will be suitable to work with a specific Lagrangian density l, which is linked with Lagrangian $\mathcal{L}$ by the relation

$$\mathcal{L} = \rho l.$$

Thus, the action functional is defined by the relation

$$\mathscr{A} = \int_{t_1}^{t_2} \int_{\mathcal{V}} \rho(\boldsymbol{x}, t) l(\boldsymbol{x}, t)\, \mathrm{d}\nu\, \mathrm{d}t. \tag{4.2}$$

As it was mentioned at the beginning of this chapter, the action functional describes the dynamics of continuum, which consists of material points $P(\boldsymbol{X}) = \boldsymbol{X}$. Each material point is moving upon its trajectory $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, \mathrm{t})$. Accordingly to the movement of the material points, there has to be an interaction between them. Due to the aspiration to take this interaction into account, the specific density l is taken as

$$l(\boldsymbol{x}, \mathrm{t}) \equiv \frac{|\boldsymbol{v}^2(\boldsymbol{x}, \mathrm{t})|}{2} - \Phi(\boldsymbol{x}) - u - \boldsymbol{X} \cdot \dot{\overline{\boldsymbol{\beta}}}, \tag{4.3}$$

where the last term $\boldsymbol{X} \cdot \dot{\overline{\boldsymbol{\beta}}}$ (i.e. the product of material points $\boldsymbol{X}$, which are passing through the geometrical point $\boldsymbol{x}$, and a *friction velocity* $\boldsymbol{\beta}$) represents the energy generated or dissipated by the friction interaction or, more precisely, the dissipation energy caused by the friction. Therefore, the friction velocity is chosen in the manner that the whole product has appropriate dimension of energy alike other members of the specific Lagrangian density. By other members are meant an internal energy u of the system, a potential of a volume forces Φ and the kinetic energy with velocity $\boldsymbol{v}$ in the Eulerian sense.

[1]This fact is commented in Seliger and Whitham [SW67].

4.1 Fluids

The thesis essentially focuses on fluids, particularly on the balances of motion related to them; more specifically, on the balances that involve the connection between the change of entropy s and the vorticity $\boldsymbol{w}$.

In the case of Thermo-viscous fluids it is reasonable to make an assumption, that the internal energy depends on the specific entropy $s = s(\boldsymbol{x}, \mathrm{t})$ and the density of mass $\rho = \rho(\boldsymbol{x}, \mathrm{t})$. According to the relation (4.3) we obtain the form of the specific Lagrangian density

$$l(\boldsymbol{x}, \mathrm{t}) \equiv \frac{|\boldsymbol{v}^2(\boldsymbol{x}, \mathrm{t})|}{2} - \Phi(\boldsymbol{x}) - u(s(\boldsymbol{x}, \mathrm{t}), \rho(\boldsymbol{x}, \mathrm{t})) - \boldsymbol{X} \cdot \dot{\overline{\boldsymbol{\beta}}}. \tag{4.4}$$

Since the motion of material points is described, it is the right moment to emphasize it using the notation including the trajectories

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t).$$

The advantage of this notation is that it points at the fact that we are „working" with the real constituents of continuum instead of the field approach of Classical continuum mechanics.

According to this notion the Lagrangian coordinates will be labeled $\boldsymbol{\chi}^{-1}$, where the notion clarifies the connection between $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^{-1}$.

Let us consider a variation[2] of the above-mentioned action functional with the trajectories

$$\delta\mathscr{A}(\boldsymbol{\chi}, \boldsymbol{\beta}, \boldsymbol{v}), \tag{4.5}$$

where the bracket $(\boldsymbol{\chi}, \boldsymbol{\beta}, \boldsymbol{v})$ denotes consideration of the trajectory, friction velocity and velocity as *independent variables* of the given variational principle, which are varied independently in order to observe the results in the shape of the balance laws.

Further we will assume that the integration area $(t_1, t_2) \times \mathcal{V}$ is fixed. From a strictly mathematical point of view, these premises are not the only ones, which is essential to achieve a *well defined* mathematical structure, i.e. that the continuity, differentiability and other necessities to all the modifications to be done make sense. The rigorous mathematical work considering the *Variational principles* is provided in a brilliant book written by Berdichevsky [Ber09].

Under these assumptions, the variational derivative has the same properties (i.e. Chain rule, Leibnitz rule, etc.) as a regular derivative and

$$\frac{\delta\gamma}{\delta\boldsymbol{\chi}} = \frac{\partial\gamma}{\partial\boldsymbol{\chi}}.$$

From this reason the notation

$$\frac{\partial\gamma}{\partial\boldsymbol{\chi}} = \nabla_{\boldsymbol{\chi}}(\gamma)$$

is applied.

[2] Let us recall that by variation the first variation is considered, see the section 2.1.

Variation

By assumption of the fixed control volume, we get the variation of action

$$\begin{aligned}\delta\mathscr{A}(\boldsymbol{\chi},\boldsymbol{\beta},\boldsymbol{v}) &= \delta\left(\int_{t_1}^{t_2}\int_{\mathcal{V}} \rho(\boldsymbol{\chi},t)l(\boldsymbol{\chi},t)\,\mathrm{d}\nu\,\mathrm{d}t\right)\\ &= \int_{t_1}^{t_2}\int_{\mathcal{V}} \delta\left(\rho(\boldsymbol{\chi},t)l(\boldsymbol{\chi},t)\right)\,\mathrm{d}\nu\,\mathrm{d}t. \end{aligned}\tag{4.6}$$

Now the integrand from the last relation will be studied. From the linearity of variation we get

$$\delta\left(\rho(\boldsymbol{\chi},t)l(\boldsymbol{\chi},t)\right) = \delta\left(\rho l\right) = \rho\delta l + l\delta\rho. \tag{4.7}$$

The variation of the first constituent is

$$\begin{aligned}\rho\delta l &= \rho\delta\left(\frac{|\boldsymbol{v}|^2}{2} - \Phi - u + \boldsymbol{\chi}\cdot\dot{\overline{\boldsymbol{\beta}}}\right) = \\ &= \rho\left(\delta\left(\frac{|\boldsymbol{v}|^2}{2}\right) - \delta\Phi - \delta u + \delta(\boldsymbol{\chi}\cdot\dot{\overline{\boldsymbol{\beta}}})\right) = \\ &= \rho\left(\boldsymbol{v}\cdot\delta\boldsymbol{v} - \delta\Phi - \delta u - \delta\boldsymbol{\chi}^{-1}\cdot\dot{\overline{\boldsymbol{\beta}}} - \boldsymbol{\chi}^{-1}\cdot\delta\dot{\overline{\boldsymbol{\beta}}}\right). \end{aligned}\tag{4.8}$$

In fact, the variation of these quantities is just the simple application of the Chain rule with the Leibnitz rule. Just for completeness, all the variations are listed below

$$\delta\rho = \delta\rho(\boldsymbol{\chi},\mathrm{t}) = \nabla_{\boldsymbol{\chi}}(\rho)\cdot\delta\boldsymbol{\chi} \tag{4.9}$$

$$\delta\Phi = \delta\Phi(\boldsymbol{\chi}) = \nabla_{\boldsymbol{\chi}}(\Phi)\cdot\delta\boldsymbol{\chi} \tag{4.10}$$

$$\delta\boldsymbol{\chi}^{-1} = \nabla_{\boldsymbol{\chi}}\left(\boldsymbol{\chi}^{-1}\right)\delta\boldsymbol{\chi} \tag{4.11}$$

$$\begin{aligned}\delta u &= \delta u(s(\boldsymbol{\chi}),\rho(\boldsymbol{\chi}))\\ &= \left(\frac{\partial u}{\partial s}\right)_\rho \frac{\partial s}{\partial\boldsymbol{\chi}}\cdot\delta\boldsymbol{\chi} + \left(\frac{\partial u}{\partial\rho}\right)_{\mathrm{s}}\frac{\partial\rho}{\partial\boldsymbol{\chi}}\cdot\delta\boldsymbol{\chi} = \\ &= \left(\left(\frac{\partial u}{\partial s}\right)_\rho \nabla_{\boldsymbol{\chi}}(s) + \left(\frac{\partial u}{\partial\rho}\right)_{\mathrm{s}}\nabla_{\boldsymbol{\chi}}(\rho)\right)\cdot\delta\boldsymbol{\chi}\end{aligned}\tag{4.12}$$

$$\delta\boldsymbol{\beta} = \delta\boldsymbol{\beta}(\boldsymbol{\chi}) = \nabla_{\boldsymbol{\chi}}(\boldsymbol{\beta})\,\delta\boldsymbol{\chi} \tag{4.13}$$

$$\delta\dot{\overline{\boldsymbol{\beta}}} = \delta\left([\boldsymbol{\beta}]_{,\mathrm{t}} + (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})(\boldsymbol{\beta})\right) \tag{4.14}$$

$$\begin{aligned}&= \delta\left([\boldsymbol{\beta}]_{,\mathrm{t}}\right) + \delta\left(\nabla_{x_i}(\boldsymbol{\beta})\,v_i\right) = \\ &= [\delta\boldsymbol{\beta}]_{,\mathrm{t}} + \nabla_{x_i}(\boldsymbol{\beta})\,\delta v_i + \nabla_{x_i}(\delta\boldsymbol{\beta})\,v_i = \\ &= [\delta\boldsymbol{\beta}]_{,\mathrm{t}} + \nabla_{\boldsymbol{x}}(\boldsymbol{\beta})\,\delta\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})(\delta\boldsymbol{\beta}).\end{aligned}\tag{4.15}$$

Here it is convenient to consider the Gibbs definition of entropy (3.38) to define the temperature T and the statical pressure p by the relations [3]

$$\left(\frac{\partial u}{\partial s}\right)_\rho = T \qquad \left(\frac{\partial u}{\partial\rho}\right)_{\mathrm{s}} = \frac{p}{\rho^2}, \tag{4.16}$$

[3]The issue of deriving these relations is not in the centre of focus of the thesis. More information about this topic is placed in the chapter 8 of Maršík [Mar99].

like in the case of the Extended Crocco's theorem, discussed in the Section 3.4.2.

However, there the similarity ends. The difference rises from the distinction between the gradient and the variational derivative. The gradient is just operator, which operates on some field quantity, but the variational derivative follows the direction of an actual particle (material point). Thus, the variational derivative and the associated definitions (4.16) have a physical interpretation - the change of internal energy caused in the sense of the relations (4.16). It implies that the relation (4.12) has the shape

$$\delta u = \left(T\nabla_{\boldsymbol{\chi}}(s) - p\nabla_{\boldsymbol{\chi}}\left(\rho^{-1}\right)\right)\cdot\delta\boldsymbol{\chi}. \tag{4.17}$$

The last constituent in the relation (4.8) can be expressed as

$$\begin{aligned}
\rho\boldsymbol{\chi}^{-1}\cdot\delta\dot{\overline{\boldsymbol{\beta}}} &= \rho\chi_i^{-1}\left(\left[\delta\beta_i\right]_{,\mathrm{t}} + \nabla_{x_j}(\beta_i)\,\delta v_j + \nabla_{x_j}(\delta\beta_i)\,v_j\right) = \\
&= \left[\left[\rho\chi_i^{-1}\delta\beta_i\right]_{,\mathrm{t}} - \left[\rho\chi_i^{-1}\right]_{,\mathrm{t}}\delta\beta_i\right] + \rho\chi_i^{-1}\left(\nabla_{x_j}(\beta_i)\,\delta(v_j)\right) + \\
&\quad + \left[\nabla_{x_j}\cdot\left(\rho(\chi_i^{-1}\delta\beta_i)v_j\right) - \nabla_{x_j}\cdot\left(\rho v_j\chi_i^{-1}\right)\delta\beta_i\right] = \\
&= \left[\rho(\boldsymbol{\chi}^{-1}\cdot\delta\boldsymbol{\beta})\right]_{,\mathrm{t}} + \nabla_{x_j}\cdot\left(\rho(\boldsymbol{\chi}^{-1}\cdot\delta\boldsymbol{\beta})v_j\right) \\
&\quad - \left([\rho]_{,\mathrm{t}} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})\right)(\boldsymbol{\chi}\cdot\delta\boldsymbol{\beta}) - \\
&\quad - \rho\left(\left[\chi_i^{-1}\right]_{,\mathrm{t}} + (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})\left(\chi_i^{-1}\right)\right)\delta\beta_i \\
&\quad + \rho\chi_i^{-1}\left(\delta\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\right)(\beta_i)\,,
\end{aligned} \tag{4.18}$$

where it was crucial to realize that $\delta\dot{\overline{\boldsymbol{\beta}}} \neq \dot{\overline{\delta\boldsymbol{\beta}}}$.

Altogether, we get the action functional

$$\begin{aligned}
\delta\mathscr{A} &= \int_{t_1}^{t_2}\int_{\mathcal{V}}(\rho\delta l + l\delta\rho)\,\mathrm{d}\nu\,\mathrm{d}t = \\
= \int_{t_1}^{t_2}\int_{\mathcal{V}}\Bigg\{&\rho\left[-\nabla_{\boldsymbol{\chi}}\left(\chi_i^{-1}\right)\dot{\overline{\beta_i}} - \nabla_{\boldsymbol{\chi}}(\Phi) - T\nabla_{\boldsymbol{\chi}}(s)\right]\cdot\delta\boldsymbol{\chi} + \\
&+ \left(l - \frac{p}{\rho}\right)\nabla_{\boldsymbol{\chi}}(\rho)\cdot\delta\boldsymbol{\chi} + \\
&+ \rho\left[\boldsymbol{v} - \chi_i^{-1}\nabla_{\boldsymbol{x}}(\beta_i)\right]\cdot\delta\boldsymbol{v} + \\
&+ \left[[\rho]_{,\mathrm{t}} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})\right](\boldsymbol{x}\cdot\delta\boldsymbol{\beta}) + \\
&+ \rho\dot{\overline{\boldsymbol{\chi}^{-1}}}\cdot\delta\boldsymbol{\beta}\Bigg\}\,\mathrm{d}\nu\,\mathrm{d}t + \\
&- \left[\int_{\mathcal{V}}\rho(\boldsymbol{\chi}^{-1}\cdot\delta\boldsymbol{\beta})\,\mathrm{d}\nu\right]_{t_1}^{t_2} - \\
&- \int_{t_1}^{t_2}\int_{\partial\mathcal{V}}\rho(\boldsymbol{\chi}^{-1}\cdot\delta\boldsymbol{\beta})\boldsymbol{v}\cdot\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t.
\end{aligned} \tag{4.19}$$

The Hamilton's theory implies that in the case of extrema, the variation of action functional has to be equal to zero.

Since the variations of friction velocity $\delta\boldsymbol{\beta}$ *vanish* on the boundary[4] of the integration area $(t_1, t_2) \times \mathcal{V}$, the last two members in the equation (4.19) identically equal zero, so they do not make any restriction to the motion of continuum.

[4] This is an analogy of the so called *fixed ends* in the classical Hamilton - Jacobi theory

Let us take a look on the remaining part of the equation. Since the variations of trajectory, velocity and friction velocity are arbitrary *in* the area of integration, the extreme will be reached if everywhere in the integration area the following is fulfilled

$$
\begin{aligned}
&\delta\boldsymbol{\chi}: && \rho\left[-\nabla_{\boldsymbol{\chi}}\left(\chi_i^{-1}\right)\dot{\overline{\beta_i}}-\nabla_{\boldsymbol{\chi}}(\Phi)-T\nabla_{\boldsymbol{\chi}}(s)\right]=0\\
&\delta\boldsymbol{\chi}: && \left(l-\frac{p}{\rho}\right)\nabla_{\boldsymbol{\chi}}(\rho)=0\\
&\delta\boldsymbol{v}: && \left[\boldsymbol{v}-\chi_i^{-1}\nabla_{\boldsymbol{x}}(\beta_i)\right]=0\\
&\boldsymbol{x}\delta\boldsymbol{\beta}: && [\rho]_{,\mathrm{t}}+\nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})=0\\
&\delta\boldsymbol{\beta}: && \rho\dot{\overline{\boldsymbol{\chi}^{-1}}}=0.
\end{aligned}
$$

The variation is achieved, the extrema is found, thus the notation including $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^{-1}$ is no further needed. Recalling that the trajectory was defined as $\boldsymbol{x}=\boldsymbol{\chi}(\boldsymbol{X},\mathrm{t})$ and its „inverse mapping" as $\boldsymbol{X}=\boldsymbol{\chi}^{-1}(\boldsymbol{x},\mathrm{t})$, the extrema conditions can be rewritten to a more suitable form

$$-\nabla_{\boldsymbol{x}}(X_i)\dot{\overline{\beta_i}}-\nabla_{\boldsymbol{x}}(\Phi)-T\nabla_{\boldsymbol{x}}(s)=0 \tag{4.20}$$

$$\left(l-\frac{p}{\rho}\right)\nabla_{\boldsymbol{x}}(\rho)=0 \tag{4.21}$$

$$\boldsymbol{v}-X_i\nabla_{\boldsymbol{x}}(\beta_i)=0 \tag{4.22}$$

$$[\rho]_{,\mathrm{t}}+\nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})=0 \tag{4.23}$$

$$\dot{\overline{\boldsymbol{X}}}=0, \tag{4.24}$$

where it assumed that the density ρ is arbitrary.

4.1.1 Interpretation of the conditions of extreme of the fluids action functional

Let us look individually at each balance equation in the context of Classical continuum mechanics and their physical meaning, when needed.

From the relation (4.23) it is seen that the requirement of the minimum of action functional implies complying with the continuity equation as well as the law of total mass conservation in the continuum mechanics.

The condition (4.24) seems to be inappropriate in the reflection of others. It includes the Lagrangian coordinate, which is not needed in the Eulerian description and in this description it indicates that the initial coordinates do not change along a trajectory. As Lin originally showed in his paper [LY87] *assuming* this condition, it is possible to obtain the rotational flow even in the case of incompressibility. He claims that the assumption of equality is justified by the requirement on the initial state to be reconstructible, whether needed or not. This is how the condition is understood.

The condition (4.22) is approached as a definition of the velocity field based on the friction and, as it will be seen, it is related to the Clebsch definition of velocity[5]. The analysis of this problematic is not the area of the main interest of the thesis, but for the completeness it is commented in the Appendix A.

The benefit of this approach when compared with the Lin's [LY87] and his followers (i.e. Seliger and Witham [SW67]) is that we are not a priori obliged to comply with this equation, but we obtain it as a result, similarly to the Clebsch definition of velocity (4.22), which will be obtained as well.

The relation (4.21) represents an energy balance and also gives us great insight into the theory. By adapting this relation we obtain that the Lagrangian density $\mathcal{L} = \rho l$ is the pressure p. It brings to light the special meaning of the pressure to the fluids and posteriorly clarifies the choice of the specific Lagrangian density. On the use of pressure as Lagrangian density the Bateman's variational principle [Bat29] is based. But one has to be careful in the case of stationary incompressibility. In that case the equality is satisfied identically and, therefore, we are not able to say anything about the energy balance and the meaning of the specific Lagrangian density l.

Further it should be said that the energy equation can be completed with a general function k, which satisfies the rule

$$\frac{\delta k}{\delta \boldsymbol{\chi}} = 0.$$

Viz the Variational principle does not essentially distinguish between the specific Lagrangian density with or without an added function k, we may take $l + k$ instead of l and the only change will appear in the energy balance. By the same modifications we obtain

$$l + k + p\rho^{-1} = 0,$$

where the constant k could be interpreted as an initial value of enthalpy h_0. Let us, for the purposes of the energy balance modification, take $k = 0$.

Further, the energy balance can be trimmed using the enthalpy definition

$$h_T = \frac{|\boldsymbol{v}|^2}{2} + \Phi + u + \frac{p}{\rho}$$

in order to highlight the link between the friction velocity and this thermodynamical potential.

It follows that

$$0 = l - \frac{p}{\rho} = \frac{|\boldsymbol{v}|^2}{2} - \Phi - u - \boldsymbol{X} \cdot \dot{\overline{\boldsymbol{\beta}}} - \frac{p}{\rho} = \tag{4.25}$$

$$= \boldsymbol{v}^2 - h_T - \boldsymbol{X} \cdot \dot{\overline{\boldsymbol{\beta}}}. \tag{4.26}$$

Thus, we get

$$\boldsymbol{v}^2 - \boldsymbol{X} \cdot \dot{\overline{\boldsymbol{\beta}}} = h_T, \tag{4.27}$$

[5] Clebsch showed in his paper [Cle09] that for the isentropic fluid flow the velocity in the shape

$$\boldsymbol{v} = \nabla (a) + b \nabla (c)$$

is the way to solve the difficulties connected with application of Eulerian description.

where the left-hand side can be modified using the Clebsch velocity definition (4.22)

$$
\begin{aligned}
&\boldsymbol{v}^2 - \boldsymbol{X} \cdot \dot{\overline{\boldsymbol{\beta}}} = \\
&= v_j v_j - X_i \dot{\overline{\beta}}_i = v_j \left(X_i \nabla_{x_j} (\beta_i) \right) - X_i \dot{\overline{\beta}}_i = \\
&= X_i \left(v_j \nabla_{x_j} (\beta_i) \right) - X_i \dot{\overline{\beta}}_i = \\
&= X_i \left(- [\beta_i]_{,\mathrm{t}} + \dot{\overline{\beta}}_i \right) - X_i \dot{\overline{\beta}}_i = \\
&= -X_i [\beta_i]_{,\mathrm{t}} = -\boldsymbol{X} \cdot [\boldsymbol{\beta}]_{,\mathrm{t}} .
\end{aligned}
$$

It implies the energy balance

$$
\boldsymbol{X} \cdot [\boldsymbol{\beta}]_{,\mathrm{t}} + h_T = 0. \tag{4.28}
$$

Let us finally take a look at the relation (4.20). From the dimensional analysis comes the meaning of the linear momentum balance

$$
\nabla_{\boldsymbol{x}} (X_i) \dot{\overline{\beta_i}} + \nabla_{\boldsymbol{x}} (\Phi) + T \nabla_{\boldsymbol{x}} (s) = 0.
$$

To derive the balance of linear momentum, which will be comparable with the so called classical theories, the relation (4.20) is obviously crucial for eliminating the nonmeasurable friction velocity $\boldsymbol{\beta}$. It is suitable to start with the energy balance (4.28). By applying a gradient on this relation we get

$$
-\nabla_{\boldsymbol{x}} (h_T) = \nabla_{\boldsymbol{x}} \left(\boldsymbol{X} \cdot [\boldsymbol{\beta}]_{,\mathrm{t}} \right) .
$$

By the Leibnitz rule the right-hand side can further be expanded

$$
\nabla_{\boldsymbol{x}} \left(\boldsymbol{X} \cdot [\boldsymbol{\beta}]_{,\mathrm{t}} \right) = \nabla_{x_j} \left(X_i [\beta_i]_{,\mathrm{t}} \right) = \nabla_{x_j} (X_i) [\beta_i]_{,\mathrm{t}} + X_i \nabla_{x_j} \left([\beta_i]_{,\mathrm{t}} \right) .
$$

Substituting the partial derivation of the friction velocity from the definition of material derivation for the right-hand side implies

$$
\nabla_{x_j} (X_i) \dot{\overline{\beta_i}} - \nabla_{x_j} (X_i) v_k \nabla_{x_k} (\beta_i) + X_i \nabla_{x_j} \left([\beta_i]_{,\mathrm{t}} \right) .
$$

Now, replacing the first constituent $\nabla_{x_j} (X_i) \dot{\overline{\beta_j}}$ from the relation (4.20) we get

$$
-\nabla_{x_j} (\Phi) - T \nabla_{x_j} (s) - \nabla_{x_j} (X_i) v_k \nabla_{x_k} (\beta_i) + X_i \nabla_{x_j} \left([\beta_i]_{,\mathrm{t}} \right) .
$$

The last constituent can be modified

$$
\begin{aligned}
X_i \nabla_{x_j} \left([\beta_i]_{,\mathrm{t}} \right) &= \left[\underbrace{\overset{(4.24)}{X_i \nabla_{x_j} (\beta_i)}}_{v_j} \right]_{,\mathrm{t}} - [X_i]_{,\mathrm{t}} \nabla_{x_j} (\beta_i) \overset{(4.22)}{=} \\
&= [v_j]_{,\mathrm{t}} + v_k \nabla_{x_k} (X_i) \nabla_{x_j} (\beta_i) .
\end{aligned}
$$

So the right-hand side has the shape

$$
-\nabla_{x_j} (\Phi) - T \nabla_{x_j} (s) + [v_j]_{,\mathrm{t}} + v_k \nabla_{x_k} (X_i) \nabla_{x_j} (\beta_i) - \nabla_{x_j} (X_i) v_k \nabla_{x_k} (\beta_i)
$$

and thus the linear momentum balance is in the shape

$$[v_j]_{,\mathrm{t}} - \left(\underline{\nabla_{x_j}(X_i)\, v_k \nabla_{x_k}(\beta_i) - v_k \nabla_{x_k}(X_i)\, \nabla_{x_j}(\beta_i)}\right) = \\ = -\nabla_{x_j}(h_T) + T\nabla_{x_j}(s) + \nabla_{x_j}(\Phi) \qquad (4.29)$$

Fortunately, the underlined constituent can be modified by the velocity field definition (4.22) and the unknown friction velocity can be eliminated entirely.

Adapting

$$\begin{aligned} &\nabla_{x_j}(X_i)\, v_k \nabla_{x_k}(\beta_i) - v_k \nabla_{x_k}(X_i)\, \nabla_{x_j}(\beta_i) = \\ =& v_k \left(\nabla_{x_j}(X_i)\, \nabla_{x_k}(\beta_i) - \nabla_{x_k}(X_i)\, \nabla_{x_j}(\beta_i)\right) \pm v_k X_i \nabla_{x_j}\left(\nabla_{x_k}(\beta_i)\right) = \\ =& v_k \left(\nabla_{x_j}\left(X_i \nabla_{x_k}(\beta_i)\right) - \nabla_{x_k}\left(X_i \nabla_{x_j}(\beta_i)\right)\right) =^{(4.22)} \\ =& v_k \left(\nabla_{x_j}(v_k) - \nabla_{x_k}(v_j)\right) = \\ =& \nabla\left(\frac{|\boldsymbol{v}|^2}{2}\right) - (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})(\boldsymbol{v}). \end{aligned}$$

and using the well known tensorial identity (3.33), we finally have the linear momentum balance

$$[\boldsymbol{v}]_{,\mathrm{t}} - \boldsymbol{v}\times\nabla_{\boldsymbol{x}}\times(\boldsymbol{v}) = -\nabla_{\boldsymbol{x}}(h_T) + T\nabla_{\boldsymbol{x}}(s) + \nabla_{\boldsymbol{x}}(\Phi), \qquad (4.30)$$

in the shape sought. When the last result is compared with the Extended Crocco's theorem (3.48)

$$[\boldsymbol{v}]_{,\mathrm{t}} - \boldsymbol{v}\times\nabla_{\boldsymbol{x}}\times(\boldsymbol{v}) = -\nabla_{\boldsymbol{x}}(h_T) + T\nabla_{\boldsymbol{x}}(s) + \rho^{-1}\nabla_{\boldsymbol{x}}\cdot\left(\mathfrak{t}^{\mathrm{dis}}\right),$$

the difference in right-hand sides is visible. This interesting fact will further be scoped.

Next an alteration of the right-hand side of the variational principle will be studied in the case of more complex materials, the Thermo-viscous-elastic materials (further solids).

4.2 Solids

The difference between the fluids and the solids from the point of view of the Variational mechanics is in the dependencies of the internal energy u. Let us recall that in the case of fluids the internal energy was considered as a function of the specific entropy s and the density ρ. In order to generalize the internal energy on the case of solids, the dependence on the density is changed to the dependence on the rate of deformation, which is represented by the *deformation* tensor $\mathfrak{e}$. Thus we have the specific Lagrangian density in the shape

$$l(\boldsymbol{x},\mathrm{t}) \equiv \frac{1}{2}\left|\boldsymbol{v}^2(\boldsymbol{x},\mathrm{t})\right| - \Phi(\boldsymbol{x}) - u(s(\boldsymbol{x},\mathrm{t}),\mathfrak{e}(\boldsymbol{x},\mathrm{t})) - \boldsymbol{X}\cdot\dot{\boldsymbol{\beta}}. \qquad (4.31)$$

As has been said, there exist many similarities between the solids and the fluids in this formalism. So the procedure by which the balances are obtained is likewise. The main and only difference arises from the variation of the internal energy

$$\delta u = \delta u(\mathrm{s},\mathfrak{e}) = \left(\left(\frac{\partial u}{\partial s}\right)_{\mathfrak{e}} \nabla_{\boldsymbol{\chi}}(s) + \left(\frac{\partial u}{\partial \mathfrak{e}}\right)_{\mathrm{s}} \nabla_{\boldsymbol{\chi}}(\mathfrak{e})\right)\delta\boldsymbol{\chi}.$$

Analogously, when the *Gibbs definition of entropy of Thermo-viscous-elastic material* is considered, it is possible to write

$$\left(\frac{\partial u}{\partial s}\right)_{\mathfrak{e}} = T \qquad \left(\frac{\partial u}{\partial \mathfrak{e}}\right)_{\mathrm{s}} = \frac{\mathfrak{t}^{\mathrm{el}}}{\rho},$$

where $\mathfrak{t}^{\mathrm{el}}$ is the *elastic part* of the stress tensor, which implies

$$\delta u = \left(T\nabla\left(s\right) + \frac{{t^{\mathrm{el}}}_{ij}}{\rho}\nabla\left(e_{ij}\right)\right)\delta\boldsymbol{\chi}. \tag{4.32}$$

Since this is the only change in the variations of the action functional, we can rewrite the relation (4.8) in the manner

$$\begin{aligned}
\rho\delta l &= \rho\left(\boldsymbol{v}\cdot\delta\boldsymbol{v} - \delta\Phi - \delta u - \delta\boldsymbol{\chi}^{-1}\cdot\dot{\overline{\boldsymbol{\beta}}} - \boldsymbol{\chi}^{-1}\cdot\delta\dot{\overline{\boldsymbol{\beta}}}\right) = \\
&= \rho\left(-\nabla_{\boldsymbol{\chi}}\left(\chi_i^{-1}\right)\dot{\overline{\beta_i}} - \nabla_{\boldsymbol{\chi}}\left(\Phi\right) - T\nabla_{\boldsymbol{\chi}}\left(s\right) - \frac{{t^{\mathrm{el}}}_{kl}}{\rho}\nabla_{\boldsymbol{\chi}}\left(e_{kl}\right)\right)\delta\boldsymbol{\chi} \\
&+ \rho\left[\boldsymbol{v} - \chi_i^{-1}\nabla_{\boldsymbol{x}}\left(\beta_i\right)\right]\cdot\delta\boldsymbol{v} + \Big[[\rho]_{,\mathrm{t}} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})\Big](\boldsymbol{x}\cdot\delta\boldsymbol{\beta}) + \\
&+ \rho\dot{\overline{\boldsymbol{\chi}^{-1}}}\cdot\delta\boldsymbol{\beta} - \left[\rho(\boldsymbol{\chi}^{-1}\cdot\delta\boldsymbol{\beta})\right]_{,\mathrm{t}} - \nabla_{\boldsymbol{x}}\left(\rho(\boldsymbol{\chi}^{-1}\cdot\delta\boldsymbol{\beta})\boldsymbol{v}\right).
\end{aligned} \tag{4.33}$$

This allows us to compose the variation of action functional

$$\begin{aligned}
\delta\mathscr{A}(\boldsymbol{\chi},\boldsymbol{\beta},\boldsymbol{v}) &= \int_{t_1}^{t_2}\int_{\mathcal{V}}\left(\rho\delta l + l\delta\rho\right)\,\mathrm{d}\nu\,\mathrm{d}t = \\
= \int_{t_1}^{t_2}\int_{\mathcal{V}}\Bigg\{\rho&\left[-\nabla_{\boldsymbol{\chi}}\left(\chi_i^{-1}\right)\dot{\overline{\beta_i}} - \nabla_{\boldsymbol{\chi}}\left(\Phi\right) - T\nabla_{\boldsymbol{\chi}}\left(s\right) - \frac{{t^{\mathrm{el}}}_{kl}}{\rho}\nabla_{\boldsymbol{\chi}}\left(e_{kl}\right)\right]\cdot\delta\boldsymbol{\chi} + \\
&+ l\nabla_{\boldsymbol{\chi}}\left(\rho\right)\delta\boldsymbol{\chi} + \rho\left[\boldsymbol{v} - \chi_i^{-1}\nabla_{\boldsymbol{x}}\left(\beta_i\right)\right]\cdot\delta\boldsymbol{v} + \\
&+ \Big[[\rho]_{,\mathrm{t}} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v})\Big](\boldsymbol{x}\cdot\delta\boldsymbol{\beta}) + \rho\dot{\overline{\boldsymbol{\chi}^{-1}}}\cdot\delta\boldsymbol{\beta}\Bigg\}\,\mathrm{d}\nu\,\mathrm{d}t + \\
&- \left[\int_{\mathcal{V}}\rho(\boldsymbol{\chi}^{-1}\cdot\delta\boldsymbol{\beta})\,\mathrm{d}\nu\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\partial\mathcal{V}}\rho(\boldsymbol{\chi}^{-1}\cdot\delta\boldsymbol{\beta})\boldsymbol{v}\cdot\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t.
\end{aligned} \tag{4.34}$$

As has been said, the variations of trajectory, velocity and friction velocity are arbitrary *in* the area of integration. So the necessary condition to extrema of the functional gives us that

$$\delta\boldsymbol{\chi}: \qquad -\nabla_{\boldsymbol{\chi}}\left(\chi_i^{-1}\right)\dot{\overline{\beta_i}} - \nabla_{\boldsymbol{\chi}}\left(\Phi\right) - T\nabla_{\boldsymbol{\chi}}\left(s\right) - \frac{{t^{\mathrm{el}}}_{kl}}{\rho}\nabla_{\boldsymbol{\chi}}\left(e_{kl}\right) = 0 \tag{4.35}$$

$$\delta\boldsymbol{\chi}: \qquad \nabla_{\boldsymbol{\chi}}\left(\rho\right)l = 0 \tag{4.36}$$

$$\delta\boldsymbol{v}: \qquad \boldsymbol{v} - \chi_i^{-1}\nabla_{\boldsymbol{x}}\left(\beta_i\right) = 0$$

$$\delta\boldsymbol{\beta}: \qquad [\rho]_{,\mathrm{t}} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v}) = 0$$

$$\delta\boldsymbol{\beta}: \qquad \dot{\overline{\boldsymbol{\chi}^{-1}}} = 0.$$

4.2.1 Interpretation of the conditions of extreme of the solids action functional

It is obvious that the conditions

$$\begin{aligned} \boldsymbol{v} - X_i \nabla_{\boldsymbol{x}} (\beta_i) &= 0 \\ [\rho]_{,\mathrm{t}} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) &= 0 \\ \dot{\boldsymbol{X}} &= 0 \end{aligned}$$

are the same as in the case of fluids. So the commentary that has already been provided remains lawful also in the case of solids.

The second condition to the extrema

$$l = 0$$

is interpreted as an energy balance and refers to the meaning of the Lagrangian in the case of Thermo-viscous-elastic material. As the pressure p has the fundamental meaning for the fluids, the energy has for the solids. Therefore, it justifies the use of this shape of the Lagrangian density, i.e. the energy per unit mass.

This energy balance will be modified to a suitable way in order to connect the meaning of friction with physical background. Assuming the total energy $\epsilon_T = \frac{1}{2}\boldsymbol{v}^2 + \Phi + u$ and the Clebsch definition of velocity (4.22), by applying the same approach as in the section 4.1.1, we are able to obtain

$$\epsilon_T = -X_i [\beta_i]_{,\mathrm{t}} , \tag{4.37}$$

which is analogous to the relation (4.21)

$$h_T = -X_i [\beta_i]_{,\mathrm{t}} .$$

It is seen that the meaning of ϵ_T in the variational mechanics of solids is the same as the meaning of the total enthalpy h_T for the fluids. The similarity of energy balances implies that the linear momentum balance for solids has the shape

$$[\boldsymbol{v}]_{,\mathrm{t}} - \boldsymbol{v} \times \nabla_{\boldsymbol{x}} \times (\boldsymbol{v}) = -\nabla_{\boldsymbol{x}} (\epsilon_T) + T \nabla_{\boldsymbol{x}} (s) + \nabla_{\boldsymbol{x}} (\Phi) + \rho^{-1} {t^{\mathrm{el}}}_{kl} \nabla_{\boldsymbol{x}} (e_{kl}) , \tag{4.38}$$

and from the comparison to the linear momentum balance for solids

$$[\boldsymbol{v}]_{,\mathrm{t}} - \boldsymbol{v} \times \nabla \times (\boldsymbol{v}) = -\nabla (\epsilon_T) + T \nabla (s) + \rho^{-1} \nabla \cdot (\mathfrak{t}) + \rho^{-1} {t^{\mathrm{el}}}_{kl} \nabla (e_{kl}) ,$$

it is seen that the difference in theories remains the same. As it will be shown, the difference is lies in the viewing of entropy.

Chapter 5

Comparison of the Classical continuum mechanics and the Variational continuum mechanics

As has been revealed, some differences have arisen in the linear momentum balances and energy balances. Let us, therefore, comment on these differences. At first we shall start with the balance of linear momentum, where the results for fluids will be considered.

5.1 Linear momentum balance

Let us recall the results.

Variational continuum mechanics

$$[\boldsymbol{v}]_{,\mathrm{t}} - \boldsymbol{v} \times \nabla_{\boldsymbol{x}} \times (\boldsymbol{v}) = -\nabla_{\boldsymbol{x}} (h_T) + T\nabla_{\boldsymbol{x}} (s) + \nabla_{\boldsymbol{x}} (\Phi) .$$

Classical continuum mechanics

$$[\boldsymbol{v}]_{,\mathrm{t}} - \boldsymbol{v} \times \nabla_{\boldsymbol{x}} \times (\boldsymbol{v}) = -\nabla_{\boldsymbol{x}} (h_T) + T\nabla_{\boldsymbol{x}} (s) + \rho^{-1}\nabla_{\boldsymbol{x}} \cdot \left(\mathrm{t}^{\mathrm{dis}}\right) .$$

From these we obtain the connection between the potential field of body forces and dissipation processes of viscous friction

$$\mathrm{t}^{\mathrm{dis}} \sim \Phi$$

in the nonstationary flow of Viscous - elastic fluid, which was, among others, presumed in Seliger and Whitham [SW67].

From another point of view, in the cases where the variation of potential Φ is negligible, it is also crucial to describe the dissipative (friction, viscous) processes. The main difference is in the viewing of entropy. In the classical theory these processes are depicted separately through the dissipative part of stress tensor $\mathrm{t}^{\mathrm{dis}}$, while the entropy caries just about internal energy and density changes; at the same time, in the Variational principles mentioned such entropy, which describes these processes itself (concretely a heat flux caused by friction), may be considered. Generally, for more complex materials Maršík [Mar99] showed[1] that the total entropy s can be rewritten into two separate parts

$$T\,\mathrm{d}s\big(\rho, \mathrm{u}, \mathsf{t}^{\mathrm{dis}}, \boldsymbol{q}\big) = T\,\mathrm{d}s_{eq}(\rho, \mathrm{u}) + T\,\mathrm{d}s_{ext}\Big(\overline{\mathsf{t}^{\mathrm{dis}}}^{\mathrm{dev}}, \mathrm{tr}\left(\mathsf{t}^{\mathrm{dis}}\right), \boldsymbol{q}\Big),$$

where s_{eq} is equilibrium entropy, which is used in the Classical continuum mechanics and s_{ext} is the entropy that describes the dissipative processes, called *extended* entropy. However, even in this extended theory are these entropies are just local variables, which guarantee local balance (equilibrium).

Deriving from this notion, the total entropy used in Variational principles, may be divided into two parts, one of which will describe the local equilibrium of the given variables s_{eq} and the second the friction processes $s_{dis} = s_{dis}\big(\rho, \mathsf{t}^{\mathrm{dis}}\big)$. By comparison with the result of the Classical continuum mechanics (3.48) the following will then be obtained

$$T\nabla_{\boldsymbol{x}}\left(s_{dis}\right) = \rho^{-1}\nabla_{\boldsymbol{x}} \cdot \left(\mathsf{t}^{\mathrm{dis}}\right) \tag{5.1}$$

and after integration along the trajectory $\boldsymbol{\chi}$, we get the formula

$$s_{dis}(\boldsymbol{x}, \mathrm{t}) - s^0_{dis} = \int_{\boldsymbol{X}}^{\boldsymbol{x}} \left(\rho T\right)^{-1} \nabla_j \cdot \left(t^{\mathrm{dis}}_{ij}(\boldsymbol{\chi}, \mathrm{t})\right)\, \mathrm{d}\chi_i \tag{5.2}$$

for the dissipative part of entropy s_{dis}, which can be interpreted as an internal entropy produced by the friction of material point.

Next these facts will be emphasized on the example of Poiseuille fluid flow through a pipe.

5.1.1 Poiseuille fluid flow

Let us consider a fluid flow described in the cylindrical polar coordinates $(\mathrm{r}, \varphi, \mathrm{z})$ with the velocity profile

$$\boldsymbol{v} = (0, 0, v_z)^T, \quad \text{where} \tag{5.3}$$

$$v_z = v_z(\mathrm{r}, \mathrm{z}) = \frac{1}{4\mu}\frac{\partial p}{\partial z}\left(r^2 - R^2\right), \tag{5.4}$$

which is given by the balance of linear momentum, i.e. the Navier-Stokes equations in a steady incompressible fluid flow limit. This shape is called *Pouisseuile velocity profile*.

The pressure is a linear function of the z coordinate

$$p(\mathrm{z}) = \frac{p_{out} - p_{in}}{L} z + p_{in} = \frac{\Delta p}{L} z + p_{in}, \tag{5.5}$$

where p_{in} is the constant pressure on the inlet and p_{out} on the outlet. Further the notation

$$v_z = \bar{v}_z(r^2 - R^2)$$

will be used while the φ coordinate will be neglected, which is possible due to the symmetry of the supposed velocity profile.

[1] His consideration was based on the Jou's work [JCVL09].

Classical continuum mechanics

Let us recall the results of the Classical continuum mechanics theory. The linear momentum balance in cylindrical polar coordinates

$$r: \qquad -v_z \frac{\partial v_z}{\partial r} = -\frac{\partial h_T}{\partial r} + T\frac{\partial s_{eq}}{\partial r} \tag{5.6}$$

$$z: \qquad 0 = -\frac{\partial h_T}{\partial z} + T\frac{\partial s_{eq}}{\partial z} + \mu\rho^{-1}\left(\frac{\partial^2 v_z}{\partial r^2} + r^{-1}\frac{\partial v_z}{\partial r}\right) \tag{5.7}$$

can be rewritten by using the total enthalpy definition in the considered limit

$$h_T = 0.5v_z^2 + c_v T + p\rho^{-1},$$

where c_v is a specific heat at a constant volume, to the form

$$r: \qquad 0 = -c_v\frac{\partial T}{\partial r} + T\frac{\partial s_{eq}}{\partial r} \tag{5.8}$$

$$z: \qquad 0 = \Delta p\rho^{-1}L^{-1} - c_v\frac{\partial T}{\partial z} + T\frac{\partial s_{eq}}{\partial z} + \mu\rho^{-1}\left(\frac{\partial^2 v_z}{\partial r^2} + r^{-1}\frac{\partial v_z}{\partial r}\right). \tag{5.9}$$

This implies the following shape of equilibrium entropy

$$s_{eq} = c_v \ln T + s_0, \tag{5.10}$$

where the temperature is a solution of the heat transfer equation (3.24).

Variational continuum mechanics

Since it is experimentally verified that the Poisseuil's flow is generally unstable (the flow out of the axis is on the limit of stability), it cannot be supposed that the flow in arbitrary interval $(0,t) \times (0.L)$ is stationary. From this reason it is not possible to assume that total enthalpy h_T is constant in the whole flow field. Its value depends on the magnitude of the friction forces[2] $[\boldsymbol{\beta}]_{,\mathrm{t}}$, which are caused by the interaction of a material point with its surroundings (in this limit by viscosity and by heat flux), as was mentioned before. From the fenomenologic point of view the rate of such energetic interaction, described by the First law of thermodynamics, in continuum mechanics is represent by the internal energy balance (3.16), in extension by the heat transfer equation (3.22).

The balance of linear momentum in the given limit is

$$r: \qquad -v_z\frac{\partial v_z}{\partial r} = -\frac{\partial h_T}{\partial r} + T\frac{\partial s}{\partial r} \tag{5.11}$$

$$z: \qquad 0 = -\frac{\partial h_T}{\partial z} + T\frac{\partial s}{\partial z}. \tag{5.12}$$

When we use the total enthalpy definition together with the shape of the equilibrium entropy (5.10), we obtain from the previous balance

$$r: \qquad 0 = T\frac{\partial s_{dis}}{\partial r} \tag{5.13}$$

$$z: \qquad 0 = -\Delta p\rho^{-1}L^{-1} + T\frac{\partial s_{dis}}{\partial z}. \tag{5.14}$$

[2] It is related with the *unreality* of stationary fluid flow. Stationary fluid flow is just a made up process, which seems to be observed on big scales.

Assuming that $T = T(\mathrm{z})$ the s_{dis} can be explicitly evaluated

$$s_{dis}(\mathrm{z}) = \Delta p \rho^{-1} L^{-1} \int_0^z T^{-1}(\zeta)\,\mathrm{d}\zeta.$$

It means that during the fluid flow the entropy s_{dis} rises with the length of the friction interaction. As an analogy to this situation a viscous flow can be used, as it is warmed up with the length of flow and is cooled by the transfer of heat through a boundary. The transfer of heat is handled by the equilibrium entropy s_{eq} and the warming is caused by s_{dis}.

5.1.2 Rigid body motion

Further, let us observe the classical mechanics limit on the derived relation (4.30), specifically the rigid body limit. By the limit an observation of fixed geometrical point $\boldsymbol{x} = \boldsymbol{X}$ in the system is meant, where both the entropy and enthalpy are constant. From the balance we thus obtain

$$\left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{\boldsymbol{X}} - \nabla\left(\Phi\right) = 0,$$

which implies that this kind of system is in equilibrium[3].

5.2 Energy balance

Next, we will focus on the derived energy balance (4.25)

$$\boldsymbol{v}^2 - \boldsymbol{X} \cdot \dot{\overline{\boldsymbol{\beta}}} = h_T = 0.5\boldsymbol{v}^2 + u(\rho, \mathrm{s}) + p\rho^{-1}.$$

In order to find the proper physical interpretation of the friction term $\boldsymbol{X} \cdot \dot{\overline{\boldsymbol{\beta}}}$, we shall replace the total enthalpy h_t in the classical energy balance equation

$$\rho \dot{\overline{h_T}} = [p]_{,\mathrm{t}} - \nabla \cdot (\boldsymbol{q}) + \nabla \cdot \left(\mathrm{t}^{\mathrm{dis}} \boldsymbol{v}\right) \tag{5.15}$$

by the alternative formula (4.25). The explicit relation for the friction introduced in the Variational principle is

$$\rho \dot{\overline{X \cdot \dot{\overline{\boldsymbol{\beta}}}}} = \rho \dot{\overline{\boldsymbol{v}^2}} - [p]_{,\mathrm{t}} + \nabla \cdot (\boldsymbol{q}) - \nabla \cdot \left(\mathrm{t}^{\mathrm{dis}} \boldsymbol{v}\right) \tag{5.16}$$

The energy (or power) of the friction force is transformed to all the energies taking place in this physical process. The equation (5.16) shows the complexity of friction processes and their influence on the local pressure change p, heat flux $\boldsymbol{q}$, velocity $\boldsymbol{v}$ and dissipative part of stress tensor $\mathrm{t}^{\mathrm{dis}}$. Friction force $\dot{\overline{\boldsymbol{\beta}}}$ is nonmeasurable quantity and its existence is demonstrated by means of entropy generation as it is indicated in the case of Poisseuile flow, which has been commented in this Chapter.

[3]This balance is a continuum mechanics equivalent of Second Newtonian law.

Chapter 6

Vorticity generation

Let us observe the influence of the entropy gradient on the vorticity, which is assured through the derived balance of linear momentum

$$\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{v} \times \nabla_{\boldsymbol{x}} \times (\boldsymbol{v}) = -\nabla_{\boldsymbol{x}} (h_T) + T \nabla_{\boldsymbol{x}} (s) + \nabla_{\boldsymbol{x}} (\Phi) ,$$

derived from the Variational principles of continuum mechanics.

6.1 Isenthalpic fluid flow

Let us recall that the total stationary limit is in general unattainable, as has been said before, so in order to neglect the influence of total enthalpy h_T such situation has to be assumed, where the constancy of total enthalpy is performed by its nature. One of such processes are flows in boundary layers with the flow of Prandtl's number equal one,viz e.g. [Sch00]. This responds to the situation, where the heat produced by viscosity is immediately led away by heat conduction. This situation occurs e.g. in the flow of wet steam, where the influence of the potential gradient is also negligible as well. In such situation

$$-\boldsymbol{v} \times \boldsymbol{w} = T \nabla_{\boldsymbol{x}} (s) , \tag{6.1}$$

so the vorticity $\boldsymbol{w}$ is generated just by entropy gradient. The same result is also obtained from the Classical continuum mechanics theory.

6.2 Suggestion to the experimental verification

The previous closure, i.e. the relation (6.1) which points at the influence of entropy gradient on the field of vorticity generation, may be confirmed experimentally. Next, the suitable form of such experiment will be commented. The reasoning following, also refers to the task of determination of the relation between the change of circulation Γ

$$\Gamma = \oint_c \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{c} \tag{6.2}$$

along closed curve c and the released heat H.

The experiment will consist of a flow around a cylinder, where the cylinder should contain some water foam jets or another evaporation device, which is able to locally

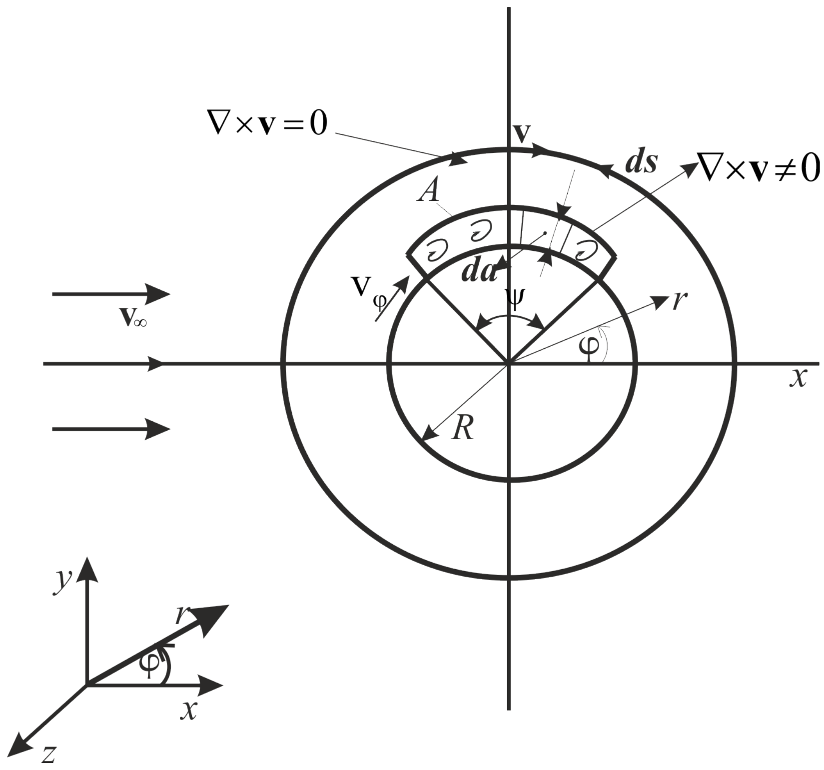

Figure 6.1: The circulation Γ generated by the velocity field $\boldsymbol{v}(\boldsymbol{x}, \mathrm{t})$ with the $\boldsymbol{w} = \nabla \times (\boldsymbol{v}) \neq 0$ induces the lift force, denoted as F_y

produce entropy gradient by a change of heat. The geometry is shown in the figure 6.1.

Next we shall make some theoretical predictions connected with the experiment. Since we are interested in the change of vorticity in the marked box, which follows the shape of cylinder, it is clear that use of curvilinear coordinates, which captures naturally the geometry, will be plausible. It obvious, that the polar cylindrical coordinates $(\mathrm{r}, \varphi, \mathrm{z})$ are the most suitable. The introduction to this type of curvilinear coordinates together with the derivations of the forms of differential operators is placed in the Appendix B.

Let us recall the vorticity defined as

$$\boldsymbol{w} = \nabla \times (\boldsymbol{v})$$

and in the curvilinear coordinates accordingly to (B.5)

$$\nabla \times (\boldsymbol{v}) = \left(r^{-1}\frac{\partial v_z}{\partial \varphi} + \frac{\partial v_\varphi}{\partial z}, \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, r^{-1}\frac{\partial r v_\varphi}{\partial r} + r^{-1}\frac{\partial v_r}{\partial \varphi} \right).$$

By the geometry displayed in the figure 6.1 it is seen that the crucial is analysis of boundary layer of the circle. Thus we are allowed to neglect the z-direction component of velocity v_z along with the derivatives in this direction. By the restriction on the boundary layer we are not concerned to observe the radial velocity component v_r, so we neglect it as well. Thus we get

$$\nabla \times (\boldsymbol{v})|_{r \in (R, R+\delta)} = \left(0, 0, \left. \frac{\partial v_\varphi}{\partial r} \right|_{r \in (R, R+\delta)} \right). \tag{6.3}$$

In the last result, slow fluid flow was also considered, so the influences of compressibility are negligible[1]. Thus

$$\boldsymbol{v} \times \nabla \times (\boldsymbol{v})|_{r\in(R,R+\delta)} = \left(v_\varphi \frac{\partial v_\varphi}{\partial r}\bigg|_{r\in(R,R+\delta)}, 0, 0 \right). \tag{6.4}$$

By the equation (6.1) the volume force on the fluid has the radial direction only.

Let us make a simplified assumption that the change of $T\nabla_r(s)$ is in the boundary layer directly proportional to the *evaporated* heat H_{evap} and indirectly to the thickness of the boundary (evaporation) layer ξ. Together it gives

$$v_\varphi \frac{\partial v_\varphi}{\partial r} \sim \frac{H_{evap}}{\xi}, \tag{6.5}$$

which implies the circulation

$$\Gamma = \int_A \boldsymbol{w} \cdot \mathrm{d}\boldsymbol{a} = \int_A \frac{\partial v_\varphi}{\partial r} \, \mathrm{d}a \sim \frac{AH_{evap}}{\xi v_\varphi}\bigg|_{A=\psi R\xi} \sim \frac{\psi R H_{evap}}{v_\varphi}, \tag{6.6}$$

where ψ is the angle on which the evaporation jets are placed.

Further can be estimated that the tangential velocity component v_φ in the evaporation volume $A\xi$ is directly proportional to the inlet velocity v_∞ by constant[2] a and that the evaporation heat H_{evap} in the case of steam is proportional to the specific evaporation heat of water H_v and the change of concentration of water in steam, called *humidity* and denoted Δc_w^{w+a}. As the denotation hints, the humidity is defined by the change of mass concentration

$$c_w^{w+a} = \frac{\rho_w}{\rho_w + \rho_a}.$$

By this assumptions we obtain

$$\Gamma \sim \frac{\psi R \Delta c_w^{w+a} H_v}{a v_\infty}, \tag{6.7}$$

whence for experiment in which R, ψ and H_v are constant, it is seen, that for a change of circulation $\mathrm{d}\Gamma$ we get

$$\mathrm{d}\Gamma = \frac{\psi R H_v}{a} \, \mathrm{d}\frac{\Delta c_w^{w+a}}{v_\infty}, \tag{6.8}$$

which is the answer on the question of circulation change induced by a change of heat in the particular situation.

[1] This remains true for fluid flow with the Mach number lesser than 0.3.

[2] It is experimentally verified that the constant a belongs to the interval $(1.41,\ 2)$.

Chapter 7

Conclusion

The thesis pointed on the close relation between the fenomenologic approach of the Classical continuum mechanics and the formulation of the Variational continuum mechanics by an appropriate choice of entropy s_{dis}. Moreover, the s_{dis} was explicitly evaluated for the case of Poiseuille fluid flow. The s_{dis} can be depicted as an internal production of the total entropy s by friction, while s_{eq} is the influx caused by local equilibrium. This was demonstrated on the case of Poisseuille's fluid flow in a pipe, even founded the explicit form of s_{dis} was revealed.

By this knowledge the influence of gradient of entropy on the vorticity generation was revealed and an experiment able to verify the circulation generation by the chemical reactions and phase transitions was designed. The actual realization of such experimental device is under prof. Maršík oversight and we hope that it will be brought to a successful end.

Further it was also shown that the variational principle considered in the case of fluids is truly related to the theory which Bateman introduced in [Bat29] and that set of conditions innovated by Seliger and Whitham [SW67] is not necessary to be a priori assumed. Actually, in the given Variational principle they are implied by fluids motion and solids motion respectively.

The variational mechanics approach was also generalized on the case of solids and it has been shown that the important role of pressure p in fluids flow is taken by energy, which is conserved during the motion, in the case of solids. This is a well known fact in the Hamilton's mechanics in conservative fields. It was also shown that the consequence between the Classical and Variational continuum mechanics remains the same also in the case of such material.

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Appendix A

Velocity field analysis

Let us observe the implications of the relation

$$\boldsymbol{v} - X_i \nabla (\beta_i) = 0,$$

which is the necessary condition for the constituent including the variation of velocity $\delta \boldsymbol{v}$ in the variation of action functional to be zero in both the considered cases. Next we will study the conclusions particularly for fluids.

By expanding the last equation we obtain

$$\boldsymbol{v} = \nabla (\boldsymbol{X} \cdot \boldsymbol{\beta}) - \beta_i \nabla (X_i) . \tag{A.1}$$

This equation can be viewed as the Clebsch [Cle09] representation of velocity field. The author introduced it for the case of isoentropic fluid flow. This concept was often used to solve the difficulties arising from using the Eulerian description. The conclusions of this interpretation were deeply examined in [SW67]. In the paper the Clebsch representation was taken as an *assumption* in order to obtain a suitable fluid description. However, in our consideration it is a result of the friction interaction term, see the section 4.1.

We are allowed to separate the velocity field on the rotational $\boldsymbol{v}^{\mathrm{rot}}$ and potential $\boldsymbol{v}^{\mathrm{pot}}$ part

$$\boldsymbol{v} = \boldsymbol{v}^{\mathrm{pot}} + \boldsymbol{v}^{\mathrm{rot}}, \tag{A.2}$$

from which, by comparison with the expanded result (A.1), we obtain

$$\boldsymbol{v}^{\mathrm{pot}} = -\nabla (\omega) \qquad \omega = -\boldsymbol{X} \cdot \boldsymbol{\beta} \tag{A.3}$$

$$\boldsymbol{v}^{\mathrm{rot}} = -\beta_i \nabla (X_i) = \nabla \times (\boldsymbol{\alpha}) , \tag{A.4}$$

where the vector potential $\boldsymbol{\alpha}$ will be determined later. From the relation (A.4) the main link between the rotational part of velocity and the friction velocity is obvious. Let us recall that the rate of change of friction is given by the balance (4.20), (4.35) respectively.

Let us derive the equation for ω. Starting with the continuity equation (3.9) rewritten to the form

$$\frac{\dot{\rho}}{\rho} = \nabla \cdot (\boldsymbol{v}) = \nabla \cdot \left(\boldsymbol{v}^{\mathrm{pot}} + \boldsymbol{v}^{\mathrm{rot}} \right) ,$$

and using the fact[1] that $\nabla \cdot (\boldsymbol{v}^{\mathrm{rot}}) = 0$ along with the velocity (A.3) potential definition, we obtain the equation

$$\Delta(\omega) = \frac{\dot{\bar{\rho}}}{\rho}. \tag{A.5}$$

Let us recall the definition of circulation Γ as a closed path integral

$$\Gamma = \oint_{\partial S} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}.$$

By the Stokes theorem

$$\oint_{\partial S} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s} = \int_S \nabla \times (\boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{a}$$

and the tensorial identity[2] $\nabla \times (\nabla(\omega)) = 0$, we get

$$\Gamma = \int_S \nabla \times (\boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{a} = \int_S \nabla \times \left(\boldsymbol{v}^{\mathrm{rot}}\right) \cdot \mathrm{d}\boldsymbol{a} = \oint_{\partial S} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s}.$$

Altogether with the definition of the vorticity density $\boldsymbol{w}$

$$\int_S \nabla \times \left(\boldsymbol{v}^{\mathrm{rot}}\right) \cdot \mathrm{d}\boldsymbol{a}, \tag{A.6}$$

we have

$$\nabla \times \left(\boldsymbol{v}^{\mathrm{rot}}\right) = \boldsymbol{w}.$$

Substituting the definition of vector potential $\boldsymbol{\alpha}$ (A.4)

$$\begin{aligned}
\nabla \times \left(\boldsymbol{v}^{\mathrm{rot}}\right) = \epsilon_{ijk} \nabla_j () \, v_k^{\mathrm{rot}} = \epsilon_{ijk} \nabla_j () \, \epsilon_{klm} \nabla_{x_l} () \, \alpha_m = & \qquad \text{(A.7)}\\
= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \nabla_j () \nabla_{x_l} () \, \alpha_m = & \qquad \text{(A.8)}\\
= \nabla_i (\nabla_j (\alpha_j)) + \Delta(\alpha_i), & \qquad \text{(A.9)}
\end{aligned}$$

which will comply with

$$\nabla \cdot (\boldsymbol{\alpha}) = 0, \tag{A.10}$$

1

$$\begin{aligned}
\nabla \cdot \left(\boldsymbol{v}^{\mathrm{rot}}\right) &= \nabla \cdot (\nabla \times (\boldsymbol{\alpha})) = \nabla_j () \, \epsilon_{jkl} \nabla_k () \, \alpha_l = \\
&= \epsilon_{jkl} \nabla_j () \nabla_k () \, \alpha_l =^{j \leftrightarrow k \,\cap\, \text{Interchangeability of derivatives}} \\
&= \epsilon_{kjl} \nabla_j () \nabla_k () \, \alpha_l \\
&= -\epsilon_{jkl} \nabla_j () \nabla_k () \, \alpha_l = -\nabla \cdot \left(\boldsymbol{v}^{\mathrm{rot}}\right) \Longrightarrow \\
\nabla \cdot \left(\boldsymbol{v}^{\mathrm{rot}}\right) &= 0
\end{aligned}$$

2

$$\begin{aligned}
\nabla \times (\nabla(\omega)) &= \epsilon_{ijk} \nabla_j () \nabla_k () \, \omega = \\
&= \epsilon_{ikj} \nabla_j () \nabla_k () \, \omega = -\epsilon_{ijk} \nabla_j () \nabla_k () \, \omega = -\nabla \times (\nabla(\omega)) \Longrightarrow \\
\nabla \times (\nabla(\omega)) &= 0
\end{aligned}$$

we obtain the relation between the vector potential $\boldsymbol{\alpha}$ and the density of vorticity

$$\boldsymbol{w} = \Delta\left(\boldsymbol{\alpha}\right). \tag{A.11}$$

The condition (A.10) is called *calibration*. The introduction of this condition initiates in the Maxwell field theory[3] as a condition to make the potential unique. In the case that the calibration condition is not fulfilled by an arbitrary vector potential $\boldsymbol{\alpha}^\star$

$$\nabla\cdot\left(\boldsymbol{\alpha}^\star\right) \neq 0,$$

we can find the vector potential $\boldsymbol{\alpha}$ which fulfills the calibration condition by finding a scalar function f as a solution to

$$\Delta\left(f\right) = \nabla\cdot\left(\boldsymbol{\alpha}^\star\right) \tag{A.12}$$

and by defining

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}^\star - \nabla\left(f\right).$$

It is seen that the vector potential defined in this manner generates the same vectorial field $\nabla\times\left(\boldsymbol{\alpha}^\star\right) = \nabla\times\left(\boldsymbol{\alpha}\right)$ by the identity $\nabla\times\left(\nabla\left(\cdot\right)\right) = 0$ and, at the same time, fulfills the calibration condition

$$\nabla\cdot\left(\boldsymbol{\alpha}\right) = \nabla\cdot\left(\boldsymbol{\alpha}^\star\right) - \underbrace{\Delta\left(f\right)}_{\text{(A.12)}\nabla\cdot\left(\boldsymbol{\alpha}^\star\right)} = 0.$$

Now we will continue with searching for the direct consequence between the derived linear momenta balance (4.30) and the mentioned density of vorticity $\boldsymbol{w}$. We shall start with adapting the necessary condition (4.20), which can be rewritten

$$-\nabla\left(X_i\right)\overline{\dot{\beta_i}} = \nabla\left(\Phi\right) + T\nabla\left(s\right).$$

Now let us focus on the left-hand side

$$-\nabla\left(X_i\right)\overline{\dot{\beta_i}} = \overline{-\nabla\left(X_i\right)\beta_i}^{\,\cdot} + \beta_i\overline{\nabla\left(X_i\right)}^{\,\cdot} = \overline{\boldsymbol{v}^{\mathrm{rot}}}^{\,\cdot} + \beta_i\overline{\nabla\left(X_i\right)}^{\,\cdot},$$

where the definition of rotational part of velocity $\boldsymbol{v}^{\mathrm{rot}}$ (A.4) was used. The last constituent can further be adjusted using the following adaptations. From material derivative,

$$\overline{\nabla_j\left(X_i\right)}^{\,\cdot} = \left[\nabla_j\left(X_i\right)\right]_{,\mathrm{t}} + v_k\nabla_k\left(\nabla_j\left(X_i\right)\right) = \tag{A.13}$$

$$= \nabla_j\left(\left[X_i\right]_{,\mathrm{t}}\right) + v_k\nabla_j\left(\nabla_k\left(X_i\right)\right) = \tag{A.14}$$

$$= \nabla_j\left(\left[X_i\right]_{,\mathrm{t}}\right) + \nabla_j\left(v_k\nabla_k\left(X_i\right)\right) - \nabla_j\left(v_k\right)\nabla_k\left(X_i\right) = \tag{A.15}$$

$$= \nabla_j\left(\left[X_i\right]_{,\mathrm{t}} + v_k\nabla_k\left(X_i\right)\right) - \nabla_j\left(v_k\right)\nabla_k\left(X_i\right), \tag{A.16}$$

[3]By Maxwell field theory is mentioned the Maxwell equations. Theirs deriving as a consequence of Variational principles framework was done among other things in the paper [SW67], written by R. L. Seliger and G. B. Whitham

in combination with the Calm condition (4.24) and the definition of $\boldsymbol{v}^{\mathrm{rot}}$ (A.4), the right-hand side is reduced to the form

$$v_k^{\mathrm{rot}} \nabla \left(v_k\right).$$

Thus, the left-hand side equals

$$\dot{\overline{\boldsymbol{v}^{\mathrm{rot}}}} + v_k^{\mathrm{rot}} \nabla \left(v_k\right).$$

Now let us focus on the material derivative of the rotational part of velocity field $\dot{\overline{\boldsymbol{v}^{\mathrm{rot}}}}$. By using the tensorial identity (3.33), we obtain

$$\dot{\overline{\boldsymbol{v}^{\mathrm{rot}}}} = \left[v_i^{\mathrm{rot}}\right]_{,\mathrm{t}} + v_j \nabla_j \left(v_i^{\mathrm{rot}}\right) = \left[v_i^{\mathrm{rot}}\right]_{,\mathrm{t}} + v_j \nabla_i \left(v_j^{\mathrm{rot}}\right) - \left(\boldsymbol{v} \times \nabla \times \left(\boldsymbol{v}^{\mathrm{rot}}\right)\right)_i .$$

Further, by the properties of velocity field $\boldsymbol{v} = \boldsymbol{v}^{\mathrm{pot}} + \boldsymbol{v}^{\mathrm{rot}}$ we shall continue adapting the left-hand side

$$\begin{aligned} &\left[v_i^{\mathrm{rot}}\right]_{,\mathrm{t}} + v_j \nabla_i \left(v_j^{\mathrm{rot}}\right) - \left(\boldsymbol{v} \times \nabla \times \left(\boldsymbol{v}^{\mathrm{rot}}\right)\right)_i + v_k^{\mathrm{rot}} \nabla_i \left(v_k\right) = \\ &= \left[v_i - v_i^{\mathrm{pot}}\right]_{,\mathrm{t}} + v_j \nabla_i \left(v_j^{\mathrm{rot}}\right) - \left(\boldsymbol{v} \times \nabla \times \left(\boldsymbol{v}\right)\right)_i + v_k^{\mathrm{rot}} \nabla_i \left(v_k\right) = \\ &= \left[v_i\right]_{,\mathrm{t}} - \left(\boldsymbol{v} \times \nabla \times \left(\boldsymbol{v}\right)\right)_i - \left[v_i^{\mathrm{pot}}\right]_{,\mathrm{t}} + v_j \nabla_i \left(v_j^{\mathrm{rot}}\right) + v_k^{\mathrm{rot}} \nabla_i \left(v_k\right). \end{aligned}$$

Now it is the proper moment to substitute the derived linear momenta balance and write the whole equation

$$\begin{aligned} -\nabla \left(h_T\right) + T \nabla \left(s\right) + \nabla \left(\Phi\right) - \left[v_i^{\mathrm{pot}}\right]_{,\mathrm{t}} + v_j \nabla_i \left(v_j^{\mathrm{rot}}\right) + v_k^{\mathrm{rot}} \nabla_i \left(v_k\right) = \\ = \nabla \left(\Phi\right) + T \nabla \left(s\right). \end{aligned}$$

It is obvious that we can continue with

$$-\left[v_i^{\mathrm{pot}}\right]_{,\mathrm{t}} + v_j \nabla_i \left(v_j^{\mathrm{rot}}\right) + v_k^{\mathrm{rot}} \nabla_i \left(v_k\right) = \nabla \left(h_T\right). \tag{A.17}$$

The definition of the potential part of velocity field and the decomposition (A.2) gives us

$$\begin{aligned} \nabla \left(h_T\right) &= \left[\nabla_i \left(\omega\right)\right]_{,\mathrm{t}} + v_j \nabla_i \left(v_j^{\mathrm{rot}}\right) + v_k^{\mathrm{rot}} \nabla_i \left(v_k\right) = \\ &= \nabla \left(\dot{\omega} - v_j \nabla_j \left(\omega\right)\right) + v_j \nabla_i \left(v_j^{\mathrm{rot}}\right) + v_k^{\mathrm{rot}} \nabla_i \left(v_k\right) = \\ &= \nabla \left(\dot{\omega}\right) + \nabla \left(v_j (v_j - v_j^{\mathrm{rot}})\right) + v_j \nabla_i \left(v_j^{\mathrm{rot}}\right) + v_k^{\mathrm{rot}} \nabla_i \left(v_k\right) = \\ &= \nabla \left(\dot{\omega}\right) + \nabla \left(\boldsymbol{v}^2\right). \end{aligned}$$

This result is obtained also from the energy balance (4.27)

$$h_T = \boldsymbol{v}^2 - \boldsymbol{X} \cdot \dot{\boldsymbol{\beta}} = \boldsymbol{v}^2 - \dot{\overline{\boldsymbol{X} \cdot \boldsymbol{\beta}}} = \boldsymbol{v}^2 + \dot{\omega}.$$

Finally it is seen that the potential part of the velocity field $\boldsymbol{v}^{\mathrm{pot}}$ and the rotational part of velocity field $\boldsymbol{v}^{\mathrm{pot}}$ influence each other.

Appendix B

Cylindrical polar coordinates

The following Appendix lists the differential operators rewritten in the cylindrical polar coordinates $(\mathrm{r}, \varphi, \mathrm{z})$, where the Landau's work [LL87] is used as a source.

Gradient

Scalar

$$\nabla\left(j\right) = \frac{\partial j}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial j}{\partial \varphi}\mathbf{e}_\varphi + \frac{\partial j}{\partial z}\mathbf{e}_z \tag{B.1}$$

Vector

$$\begin{aligned}\nabla\left(\boldsymbol{j}\right) = {} & \frac{\partial j_r}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{1}{r}\left(\frac{\partial j_r}{\partial \varphi} - j_\varphi\right)\mathbf{e}_r \otimes \mathbf{e}_\varphi + \frac{\partial j_r}{\partial z}\mathbf{e}_r \otimes \mathbf{e}_z \\ & + \frac{\partial j_\varphi}{\partial r}\mathbf{e}_\varphi \otimes \mathbf{e}_r + \frac{1}{r}\left(\frac{\partial j_\varphi}{\partial \varphi} + j_r\right)\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \frac{\partial j_\varphi}{\partial z}\mathbf{e}_\varphi \otimes \mathbf{e}_z \\ & + \frac{\partial j_z}{\partial r}\mathbf{e}_z \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial j_z}{\partial \varphi}\mathbf{e}_z \otimes \mathbf{e}_\varphi + \frac{\partial j_z}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_z\end{aligned} \tag{B.2}$$

Divergence

Vector

$$\nabla \cdot \left(\boldsymbol{j}\right) = \frac{\partial j_r}{\partial r} + r^{-1}\left(\frac{\partial j_\varphi}{\partial \varphi} + j_r\right) + \frac{\partial j_z}{\partial z}. \tag{B.3}$$

Tensor

$$\begin{aligned}\nabla \cdot (\mathrm{j}) = {} & \frac{\partial j_{rr}}{\partial r}\boldsymbol{e}_r + \frac{\partial j_{r\varphi}}{\partial r}\boldsymbol{e}_\varphi + \frac{\partial j_{rz}}{\partial r}\boldsymbol{e}_z + \\ & + \frac{1}{r}\left[\frac{\partial j_{\varphi r}}{\partial \varphi} + (j_{rr} - j_{\varphi\varphi})\right]\boldsymbol{e}_r + \frac{1}{r}\left[\frac{\partial j_{\varphi\varphi}}{\partial \varphi} + (j_{r\varphi} + j_{\varphi r})\right]\boldsymbol{e}_\varphi + \\ & + \frac{1}{r}\left[\frac{\partial j_{\varphi z}}{\partial \varphi} + j_{rz}\right]\boldsymbol{e}_z + \frac{\partial j_{zr}}{\partial z}\boldsymbol{e}_r + \frac{\partial j_{z\varphi}}{\partial z}\boldsymbol{e}_\varphi + \frac{\partial j_{zz}}{\partial z}\boldsymbol{e}_z,\end{aligned} \tag{B.4}$$

Rotation

Vector

$$\nabla \times (\boldsymbol{v}) = \left(r^{-1} \frac{\partial v_z}{\partial \varphi} + \frac{\partial v_\varphi}{\partial z}, \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, r^{-1} \frac{\partial r v_\varphi}{\partial r} + r^{-1} \frac{\partial v_r}{\partial \varphi} \right) \tag{B.5}$$

Laplace

Scalar

$$\Delta (f) = r^{-1} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + r^{-2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}. \tag{B.6}$$

B.1 Dissipative part of the Newtonian stress tensor

The dissipation part of the stress tensor in the case of compressible Newtonian fluids is defined by the relation

$$\mathsf{t}^{\mathrm{dis}} = \lambda \nabla \cdot (\boldsymbol{v}) \, \mathbb{I} + \mu \left(\nabla (\boldsymbol{v}) + \nabla^T (\boldsymbol{v}) \right)$$

Thus in the cylindrical polar coordinates

$$t_{rr}^{\mathrm{dis}} = 2\mu \frac{\partial v_r}{\partial r} + \lambda \left(\frac{\partial v_r}{\partial r} + r^{-1} \left(\frac{\partial v_\varphi}{\partial \varphi} + v_r \right) + \frac{\partial v_z}{\partial z} \right) \tag{B.7}$$

$$t_{\varphi\varphi}^{\mathrm{dis}} = 2\mu \left(r^{-1} \frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} \right) + \lambda \left(\frac{\partial v_r}{\partial r} + r^{-1} \left(\frac{\partial v_\varphi}{\partial \varphi} + v_r \right) + \frac{\partial v_z}{\partial z} \right) \tag{B.8}$$

$$t_{zz}^{\mathrm{dis}} = 2\mu \frac{\partial v_z}{\partial z} + \lambda \left(\frac{\partial v_r}{\partial r} + r^{-1} \left(\frac{\partial v_\varphi}{\partial \varphi} + v_r \right) + \frac{\partial v_z}{\partial z} \right) \tag{B.9}$$

$$t_{r\varphi}^{\mathrm{dis}} = \mu \left(r^{-1} \frac{\partial v_r}{\partial \varphi} + \frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r} \right) \tag{B.10}$$

$$t_{\varphi z}^{\mathrm{dis}} = \mu \left(r^{-1} \frac{\partial v_z}{\partial \varphi} + \frac{\partial v_\varphi}{\partial z} \right) \tag{B.11}$$

$$t_{zr}^{\mathrm{dis}} = \mu \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right). \tag{B.12}$$